Control of the half-heat equation

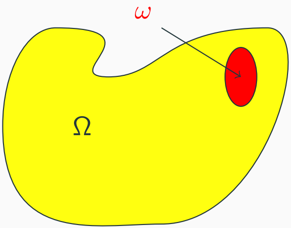
Joint work with Andreas Hartmann

Armand Koenig
Institut de Mathématiques de Bordeaux

September 26th, 2024

Inverse problem on manifolds and control

Introduction

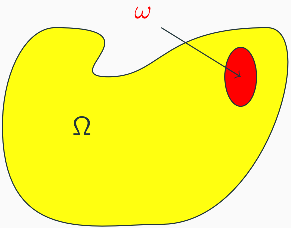


Theorem (Null-controllability of the heat equation (Lebeau & Robbiano 1995, Fursikov & Imanuvilov 1996))

For every $T > 0$ and every initial condition f_0, there exists $u \in L^2((0,T) \times \omega)$ such that the solution f of

$$(\partial_t - \Delta) f(t,x) = \mathbf{1}_{\omega} u(t,x), \quad f(0,\cdot) = f_0$$

satisfies $f(T,\cdot) = 0$.



Theorem (Null-controllability of the heat equation (Lebeau & Robbiano 1995, Fursikov & Imanuvilov 1996))

For every $T > 0$ and every initial condition f_0, there exists $u \in L^2((0,T) \times \omega)$ such that the solution f of

$$(\partial_t - \Delta)f(t,x) = \mathbf{1}_\omega u(t,x), \quad f(0,\cdot) = f_0$$

satisfies $f(T,\cdot) = 0$.

Observability

Equivalent dual problem to null-controllability:

$$(\partial_t - \Delta)g = 0 \implies \|g(T,\cdot)\|_{L^2(\Omega)} \leq C\|g\|_{L^2([0,T]\times\omega)}$$

Definition (Half-heat equation)

If $f(x) = \sum_{n\in\mathbb{Z}} a_n e^{inx}$, $|D_x| f(x) = \sum_{n\in\mathbb{Z}} |n| a_n e^{inx}$.

$$(\partial_t + |D_x|) f = \mathbf{1}_\omega u.$$

Question

- Study the control properties of the half-heat equation.
- Characterize the initial states that can be steered to 0.

Definition (Half-heat equation)

If $f(x) = \sum_{n\in\mathbb{Z}} a_n e^{inx}$, $|D_x|f(x) = \sum_{n\in\mathbb{Z}} |n| a_n e^{inx}$.

$$(\partial_t + |D_x|)f = 1_{\omega} u.$$

Question

- Study the control properties of the half-heat equation.
- Characterize the initial states that can be steered to 0.

Plan

Results

Theorem (K, 2015)

Let ω be a strict interval of $\mathbb{T}$. The control system $(\partial_t + |D_x|)f(t,x) = \mathbf{1}_\omega u(t,x)$ is not null-controllable.

Proof.
Solutions of $(\partial_t + |D_x|)g = 0$: $g(t,x) = \sum_n a_n e^{-|n|t} e^{inx}$.

Null-controllability $\implies \displaystyle\sum_{n>0} |a_n|^2 e^{-2nT} \leq C \int_{[0,T]\times\omega} \Big|\sum_{n>0} a_n e^{-nt} e^{inx}\Big|^2 \,\mathrm{d}t\,\mathrm{d}y$

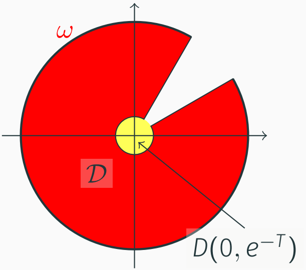
Theorem (K, 2015)

Let ω be a strict interval of $\mathbb{T}$. The control system $(\partial_t + |D_x|)f(t,x) = \mathbf{1}_\omega u(t,x)$ is not null-controllable.

Proof.
Solutions of $(\partial_t + |D_x|)g = 0$: $g(t,x) = \sum_n a_n e^{-|n|t} e^{inx}$.

Null-controllability $\implies \displaystyle\sum_{n>0} |a_n|^2 e^{-2nT} \leq C \int_{[0,T]\times\omega} \Big|\sum_{n>0} a_n e^{-nt} e^{inx}\Big|^2 \,\mathrm{d}t\,\mathrm{d}y$

- Change of variables: $z = e^{-t+ix}$
- Null-controllability $\implies$ for every polynonials $p \in \mathbb{C}[X]$, $\|p\|_{L^2(D(0,e^{-T}))} \leq C\|p\|_{L^2(\mathcal{D})}$
- This inequality does not hold.



An intermediate problem

Definition (Riesz projection)

if $f(x) = \sum_{n\in\mathbb{Z}} a_n e^{inx}$, $P_+ f(x) = \sum_{n\geq 0} a_n e^{inx}$.

Definition (The H^2 control system)

$$(\partial_t + |D_x|) f(t,x) = \mathbf{1}_\omega u(t,x), \qquad f(0,\cdot) \in L^2(\mathbb{T}) \tag{E_{L^2}}$$

Definition (Riesz projection)

if $f(x) = \sum_{n\in\mathbb{Z}} a_n e^{inx}$, $P_+ f(x) = \sum_{n\geq 0} a_n e^{inx}$.

Definition (The H^2 control system)

$$(\partial_t + |D_x|) f(t,x) = \mathbf{1}_\omega u(t,x), \qquad f(0,\cdot) \in L^2(\mathbb{T}) \qquad (E_{L^2})$$

$$(\partial_t + |D_x|) f(t,x) = P_+ \mathbf{1}_\omega u(t,x), \qquad f(0,\cdot) \in H^2(\mathbb{T}) \qquad (E_{H^2})$$

Annoyance: $H^2(\mathbb{T}) = P_+(L^2(\mathbb{T}))$ is the Hardy space

Definition

$\mathcal{NC}_{L^2}(\omega, T) = \{f_0 \in L^2(\mathbb{T}),\ \exists u \in L^2([0,T]\times\omega),$ solution f of (E_{L^2}) s.t. $f(T,\cdot) = 0\}$

$\mathcal{NC}_{H^2}(\omega, T) = \{f_0 \in H^2(\mathbb{T}),\ \exists u \in L^2([0,T]\times\omega),$ solution f of (E_{H^2}) s.t. $f(T,\cdot) = 0\}$

Results

Theorem (Hartmann-K 2024)

$\mathcal{NC}_{H^2}(\omega, T)$ and $\mathcal{NC}_{L^2}(\omega, T)$ do not depend on T.

Theorem (Hartmann-K 2024)

If $f_0 \in H^2(\mathbb{T})$ is nonzero and analytic on $\mathbb{T}$, $f_0 \notin \mathcal{NC}_{H^2}(\omega)$.

If $f_0 \in L^2(\mathbb{T})$ is nonzero and analytic on $\mathbb{T}$, $f_0 \notin \mathcal{NC}_{L^2}(\omega)$.

Theorem (Hartmann-K 2024)

$\mathcal{NC}_{H^2}(\omega)$ (resp. $\mathcal{NC}_{L^2}(\omega)$) and its complement are dense in $H^2(\mathbb{T})$ (resp. $L^2(\mathbb{T})$).

Results

Theorem (Hartmann-K 2024)

$\mathcal{NC}_{H^2}(\omega, T)$ and $\mathcal{NC}_{L^2}(\omega, T)$ do not depend on T.

Theorem (Hartmann-K 2024)

If $f_0 \in H^2(\mathbb{T})$ is nonzero and analytic on $\mathbb{T}$, $f_0 \notin \mathcal{NC}_{H^2}(\omega)$.

If $f_0 \in L^2(\mathbb{T})$ is nonzero and analytic on $\mathbb{T}$, $f_0 \notin \mathcal{NC}_{L^2}(\omega)$.

Theorem (Hartmann-K 2024)

$\mathcal{NC}_{H^2}(\omega)$ (resp. $\mathcal{NC}_{L^2}(\omega)$) and its complement are dense in $H^2(\mathbb{T})$ (resp. $L^2(\mathbb{T})$).

Theorem (Hartmann-K 2024)

Let $f_0 \in H^2(\mathbb{T})$ such that $W^{1/2,2}(\mathbb{T})$. Then $f_0 \in \mathcal{NC}_{H^2}(\omega)$ if and only if there exists $h \in W^{1/2,2}_{00}(\omega)$ such that $f_0 = P_+ h$.

$W^{1/2}_{00}(\omega) \subset \mathcal{NC}_{L^2}(\omega)$.

Internal control vs shaped control

	Internal controls	Shaped controls [Micu-Zuazua 2006] $\forall\epsilon>0, \lvert\widehat{h}(n)\rvert \geq ce^{-\epsilon n}$	Shaped controls [Micu-Zuazua 2006] $\exists\epsilon>0, \lvert\widehat{h}(n)\rvert \leq Ce^{-\epsilon n}$
Control system	$(\partial_t + \lvert D_x\rvert)f = \mathbf{1}_\omega u$	$(\partial_t + \lvert D_x\rvert)f = h(x)u(t)$	
Null-controllability	Not null-controllable	Not null-controllable	

Internal control vs shaped control

	Internal controls	Shaped controls [Micu-Zuazua 2006] $\forall\epsilon>0, \lvert\widehat{h}(n)\rvert \geq ce^{-\epsilon n}$	Shaped controls [Micu-Zuazua 2006] $\exists\epsilon>0, \lvert\widehat{h}(n)\rvert \leq Ce^{-\epsilon n}$
Control system	$(\partial_t + \lvert D_x\rvert)f = 1_\omega u$	$(\partial_t + \lvert D_x\rvert)f = h(x)u(t)$	
Null-controllability	Not null-controllable	Not null-controllable	
Set of null-controllable states	dense subspace Independant of time	$\{0\}$	$\neq \{0\}$ depends on time?
Regularity of null–controllable states	cannot be analytical		some analytical states

Control of positive frequencies

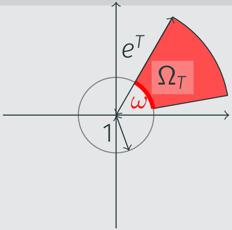
Further annoyances

$$\mathbb{R}/2\pi\mathbb{Z} = \mathbb{T} \approx \{z \in \mathbb{C},\ |z| = 1\}, \quad H^2(\mathbb{T}) \approx \{\sum_{n\geq 0} a_n z^n,\ (a_n) \in \ell^2\}, \quad f(x) \approx f(e^{ix})$$

Proposition (observability inequality)

Let $f_0 \in H^2(\mathbb{T})$. $f_0 \in \mathcal{NC}_{H^2}(\omega)$ if and only if there exists $C > 0$ such that

$$\forall p \in \mathbb{C}[X],\ \int_0^{2\pi} p(e^{it})\overline{f_0(e^{it})}\,\mathrm{d}t \leq C\|p\|_{L^2(\Omega_T)}.$$



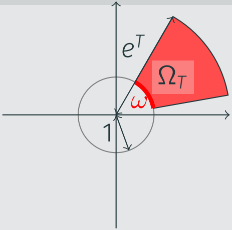
Further annoyances

$\mathbb{R}/2\pi\mathbb{Z} = \mathbb{T} \approx \{z \in \mathbb{C},\ |z| = 1\}, \quad H^2(\mathbb{T}) \approx \{\sum_{n\geq 0} a_n z^n,\ (a_n) \in \ell^2\}, \quad f(x) \approx f(e^{ix})$

Proposition (observability inequality)

Let $f_0 \in H^2(\mathbb{T})$. $f_0 \in \mathcal{NC}_{H^2}(\omega)$ if and only if there exists $C > 0$ such that

$$\forall p \in \mathbb{C}[X],\ \int_0^{2\pi} p(e^{it})\overline{f_0(e^{it})}\,\mathrm{d}t \leq C\|p\|_{L^2(\Omega_T)}.$$



Proof.

$f_0 \in \mathcal{NC}_{H^2}(\omega)$ if and only if there exists $C > 0$ such that for every $g_0 \in H^2(\mathbb{T})$,

$$\langle f_0, e^{-T|D_x|}g_0\rangle_{H^2(\mathbb{T})} \leq C\int_0^T\int_\omega |e^{-t|D_x|}g_0(e^{ix})|^2\,\mathrm{d}t\,\mathrm{d}y.$$

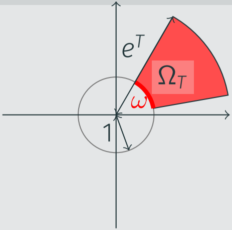
Further annoyances

$\mathbb{R}/2\pi\mathbb{Z} = \mathbb{T} \approx \{z \in \mathbb{C},\ |z| = 1\}, \quad H^2(\mathbb{T}) \approx \{\sum_{n\geq 0} a_n z^n,\ (a_n) \in \ell^2\}, \quad f(x) \approx f(e^{ix})$

Proposition (observability inequality)

Let $f_0 \in H^2(\mathbb{T})$. $f_0 \in \mathcal{NC}_{H^2}(\omega)$ if and only if there exists $C > 0$ such that

$$\forall p \in \mathbb{C}[X],\ \int_0^{2\pi} p(e^{it})\overline{f_0(e^{it})}\,\mathrm{d}t \leq C\|p\|_{L^2(\Omega_T)}.$$



Proof.
$f_0 \in \mathcal{NC}_{H^2}(\omega)$ if and only if there exists $C > 0$ such that for every $g_0 \in H^2(\mathbb{T})$,

$$\langle f_0, g_0(e^{-T\cdot})\rangle_{H^2(\mathbb{T})} \leq C \int_0^T \int_\omega |g_0(e^{-t+ix})|^2 \,\mathrm{d}t\,\mathrm{d}y.$$

$e^{-t|D_x|}g_0(e^{ix}) = \sum_{n\geq 0} \widehat{g_0}(n)e^{iny-nt} = \sum_{n\geq 0} \widehat{g_0}(n)(e^{iy-t})^n$.
Change of variables $z = e^{-t+ix}$.

Theorem (Hartmann-K 2024)

If $h \in W^{1/2,2}_{00}(\omega)$, $P_+h \in \mathcal{NC}_{H^2}(\omega)$.

Proof.

- extends h by 0 on $\partial\Omega_T$
- $h \in W^{1/2,2}(\partial\Omega_T)$
- h extends as an $W^{1,2}(\Omega_T)$ function.

Theorem (Hartmann-K 2024)

If $h \in W^{1/2,2}_{00}(\omega)$*,* $P_+ h \in \mathcal{NC}_{H^2}(\omega)$*.*

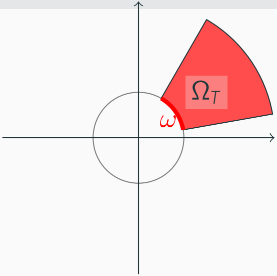
Proof.

- extends h by 0 on $\partial\Omega_T$
- $h \in W^{1/2,2}(\partial\Omega_T)$
- h extends as an $W^{1,2}(\Omega_T)$ function.
- Apply stokes formula:

$$\int_{\partial\Omega_T} \overline{h}(z)p(z)\,\mathrm{d}z = 2i\int_{\Omega_T} \partial_{\overline{z}}(\overline{h}p)(z)\,\mathrm{d}A(z)$$

Theorem (Hartmann-K 2024)

If $h \in W^{1/2,2}_{00}(\omega)$, $P_+ h \in \mathcal{NC}_{H^2}(\omega)$.



Proof.

- extends h by 0 on $\partial\Omega_T$
- $h \in W^{1/2,2}(\partial\Omega_T)$
- h extends as an $W^{1,2}(\Omega_T)$ function.
- Apply stokes formula:

$$\int_{\partial\Omega_T} \bar{h}(z)p(z)\,\mathrm{d}z = 2i\int_{\Omega_T} \partial_{\bar{z}}(\bar{h}p)(z)\,\mathrm{d}A(z)$$

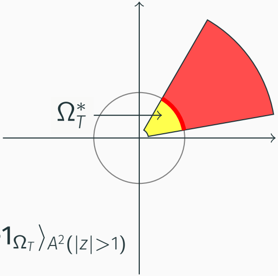
- Left-hand side: $\approx \langle h, p\rangle_{H^2} = \langle h, P_+ p\rangle_{H^2} = \langle P_+ h, p\rangle_{H^2}$.
- Right-hand side: $\int_{\Omega_T} \partial_{\bar{z}}(\bar{h}p)(z)\,\mathrm{d}A(z) = \int_{\Omega_T} p(z)\underbrace{\partial_{\bar{z}}\bar{h}(z)}_{\in L^2(\Omega_T)}\,\mathrm{d}A(z)$ $\square$

Theorem (Hartmann-K 2024)

Let $f_0 \in H^2(\mathbb{T})$. If $f_0 \in \mathcal{NC}_{H^2}(\omega)$, f_0 extends as a holomorphic function to $\mathbb{C} \setminus \mathrm{closure}(\omega)$, and $f_0(z) \xrightarrow[|z|\to\infty]{} 0$. Moreover $\int_{|z|>1} |f_0'(z)|^2 \,\mathrm{d}A(z) < +\infty$.

Proof.

- let $k_u(z) = \dfrac{1}{2\pi(1-\overline{u}z)}$. If $f_0 \in H^2(\mathbb{T})$, $f_0(u) = \langle f_0, k_u\rangle_{H^2}$.
- for all $p \in \mathbb{C}[X]$, $\ell(p) := \langle f_0, p\rangle_{H^2} \leq C\|p\|_{L^2(\Omega_T)}$
- extend ℓ by density to $A^2(\Omega_T)$ (space of holomorphic functions on Ω_T which are $L^2(\Omega_T)$)
  We can define $\ell(k_u)$ if $u \notin \Omega_T^*$.
- $\phi(u) = \begin{cases} f_0(u) & \text{if } |u| < 1 \\ \ell(k_u) & \text{if } u \notin \Omega_T^* \end{cases}$ extends f_0.
- $\|k_u\|_{L^2(\Omega_T)} = O(1/u)$: $f_0(z) \xrightarrow[|z|\to\infty]{} 0$
- $\ell(p) = \langle \varphi, p\rangle_{A^2(\Omega_T)} \quad f_0'(u) = \ell(\partial_{\overline{u}} k_u) = \langle k_u^{\mathrm{Bergman}}, \varphi \mathbf{1}_{\Omega_T}\rangle_{A^2(|z|>1)}$



Theorem (Hartmann-K 2024)

If $f_0 \in H^2(\mathbb{T}) \cap W^{1/2,2}(\mathbb{T})$, and $f_0 \in \mathcal{NC}_{H^2}(\omega)$, there exists $h \in W^{1/2,2}_{00}(\omega)$ such that $f_0 = P_+ h$.

Proof.

- Extend f_0 holomorphically to $\mathbb{C} \setminus \operatorname{closure}(\omega)$ as above
- Set
$$h(e^{i\theta}) = \lim_{r \to 1^-} \Big(f_0(re^{i\theta}) - f_0(r^{-1}e^{i\theta}) \Big)$$
- Idea: Positive frequencies of h: $f_0|_{|z|<1}$
  Negative frequencies of h: $f_0|_{|z|>1}$ $\square$

Our most precise results

Theorem (Hartmann-K 2024)

Let $f_0 \in \mathcal{NC}_{H^2}(\omega)$. There exists $h \in L^2(\mathbb{T})$ such that

- $f_0 = P_+ h$
- $\mathrm{Supp}(h) \subset \mathrm{closure}(\omega)$
- $\sum_{n \leq 0} |n| |\widehat{h}(n)|^2 < +\infty$

Theorem (Hartmann-K 2024)

Let $h \in L^2(\mathbb{T})$ such that

- $\mathrm{Supp}(h) \subset \mathrm{closure}(\omega)$
- $\sum_{n \leq 0} |n| |\widehat{h}(n)|^2 < +\infty$
- $\displaystyle\int_\omega \frac{1}{\mathrm{d}(z, \partial\omega)} |h(z)|^2 |\mathrm{d}z| < +\infty.$

Then $P_+ h \in \mathcal{NC}_{H^2}(\omega)$.

Our most precise results

Theorem (Hartmann-K 2024)

Let $f_0 \in \mathcal{NC}_{H^2}(\omega)$. *There exists* $h \in L^2(\mathbb{T})$ *such that*

- $f_0 = P_+ h$
- $\mathrm{Supp}(h) \subset \mathrm{closure}(\omega)$
- $\sum_{n \leq 0} |n| |\widehat{h}(n)|^2 < +\infty$

Theorem (Hartmann-K 2024)

Let $h \in L^2(\mathbb{T})$ *such that*

- $\mathrm{Supp}(h) \subset \mathrm{closure}(\omega)$
- $\sum_{n \leq 0} |n| |\widehat{h}(n)|^2 < +\infty$
- $\displaystyle\int_\omega \frac{1}{\mathrm{d}(z, \partial\omega)} |h(z)|^2 |\mathrm{d}z| < +\infty.$

Then $P_+ h \in \mathcal{NC}_{H^2}(\omega)$.

Theorem (Hartmann-K 2024)

Let $f_0 \in L^2(\mathbb{T})$. $f_0 \in \mathcal{NC}_{H^2}(\omega)$ *if and only if there exists* $h \in L^2(\mathbb{T})$ *such that*

- $f_0 = P_+ h$
- $\mathrm{Supp}(h) \subset \mathrm{closure}(\omega)$
- *The solution* u *of* $\begin{cases} \Delta u(x) = 0, & x \in \Omega_T \\ u(x) = h(x), & x \in \omega \\ u(x) = 0, & x \in \partial\Omega_T \setminus \omega \end{cases}$

  satisfies $\partial_z u \in L^2(\Omega_T)$

Control of all frequencies

From partial control to full control

- We know when the projection on positive frequencies is null-controllable
- If $f_0 \in \mathcal{NC}_{L^2}(\omega, T)$, $P_+ f_0 \in \mathcal{NC}_{H^2}(\omega, T)$
- Negative frequencies: just consider $\overline{f_0}$

From partial control to full control

- We know when the projection on positive frequencies is null-controllable
- If $f_0 \in \mathcal{NC}_{L^2}(\omega, T)$, $P_+ f_0 \in \mathcal{NC}_{H^2}(\omega, T)$
- Negative frequencies: just consider $\overline{f_0}$

Theorem (Hartmann-K 2024)

$$f_0 \in \mathcal{NC}_{L^2}(\omega, T) \implies \begin{cases} P_+ f_0 \in \mathcal{NC}_{H^2}(\omega) \\ P_+ \overline{f_0} \in \mathcal{NC}_{H^2}(\omega) \end{cases}$$

From partial control to full control

- We know when the projection on positive frequencies is null-controllable
- If $f_0 \in \mathcal{NC}_{L^2}(\omega, T)$, $P_+ f_0 \in \mathcal{NC}_{H^2}(\omega, T)$
- Negative frequencies: just consider $\overline{f_0}$

Theorem (Hartmann-K 2024)

$$f_0 \in \mathcal{NC}_{L^2}(\omega, T) \Longleftrightarrow \begin{cases} P_+ f_0 \in \mathcal{NC}_{H^2}(\omega) \\ P_+ \overline{f_0} \in \mathcal{NC}_{H^2}(\omega) \end{cases}$$

- We know when the projection on positive frequencies is null-controllable
- If $f_0 \in \mathcal{NC}_{L^2}(\omega, T)$, $P_+ f_0 \in \mathcal{NC}_{H^2}(\omega, T)$
- Negative frequencies: just consider $\overline{f_0}$

Theorem (Hartmann-K 2024)

$$f_0 \in \mathcal{NC}_{L^2}(\omega, T) \Longleftrightarrow \begin{cases} P_+ f_0 \in \mathcal{NC}_{H^2}(\omega) \\ P_+ \overline{f_0} \in \mathcal{NC}_{H^2}(\omega) \end{cases}$$

Proof

- Solution of half-heat: sum of holomorphic and anti-holomorphic functions in e^{-t+ix}.
- Need: recover the holomorphic part with only information on ω
- Need: continuous projection $\{f \in L^2(\Omega_T),\ \text{harmonic}\} \to A^2(\Omega_T)$.
- This already exists (Friedrichs, 1937)

- We know when the projection on positive frequencies is null-controllable
- If $f_0 \in \mathcal{NC}_{L^2}(\omega, T)$, $P_+ f_0 \in \mathcal{NC}_{H^2}(\omega, T)$
- Negative frequencies: just consider $\overline{f_0}$

Theorem (Hartmann-K 2024)

$$f_0 \in \mathcal{NC}_{L^2}(\omega, T) \Longleftrightarrow \begin{cases} P_+ f_0 \in \mathcal{NC}_{H^2}(\omega) \\ P_+ \overline{f_0} \in \mathcal{NC}_{H^2}(\omega) \end{cases}$$

Proof

- Solution of half-heat: sum of holomorphic and anti-holomorphic functions in e^{-t+ix}.
- Need: recover the holomorphic part with only information on ω
- Need: continuous projection $\{f \in L^2(\Omega_T),\ \text{harmonic}\} \to A^2(\Omega_T)$.
- This already exists (Friedrichs, 1937)

Translate properties of the H^2 control system to the L^2 control system

That’s all folks!