Control of the half-heat equation

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Abstract

We investigate the half heat equation controlled from a sub-arc on the unit circle. The half heat equation marks the boundary case in the range of general fractional heat equations between null-controllability and lack of null-controllability. As it turns out, complex function theory and spaces of holomorphic functions show to be powerful tools in this situation. In particular, null controllable functions can be described translating first the problem to positive frequencies where Hardy, Bergman and Dirichlet spaces come into play. The case of general L^2-data follows then by classical results on decomposition of harmonic functions on suitable domains into the sum of functions in a Bergman space and its conjugate. A striking fact is that it is possible to show time-independence of null controllable states using so-called separation of singularities without an explicit characterization of these states.

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1 Introduction

1.1 Setting

Let $\mathbb{T}$ be the unit complex circle. Functions in $L^2(\mathbb{T})$ can be written as $f(\mathrm{e}^{\mathrm{i}x}) = \sum_{n\in\mathbb{Z}} \widehat{f}(n)\mathrm{e}^{inx}$, where $(\widehat{f}(n))_n \in \ell^2(\mathbb{Z})$. With this notation, we will frequently identify $\mathbb{T}$ and $\mathbb{R}/2\pi\mathbb{Z} \sim [0, 2\pi)$ endowed with the Lebesgue measure (with total measure 2π). We will use in a similar fashion the words arc and interval for a connected subset of $\mathbb{T}$. Let $|D|$ be the unbounded operator on $L^2(\mathbb{T})$ defined by

$$|D|f(\mathrm{e}^{\mathrm{i}x}) = \sum_{n\in\mathbb{Z}} |n|\widehat{f}(n)\mathrm{e}^{inx}. \tag{1}$$

Given an open subset $\omega \subset \mathbb{T}$, we consider the control system (that we will call "L^2 control system"):

$$\begin{cases} (\partial_t + |D|)f(t,\mathrm{e}^{\mathrm{i}x}) = \mathbb{1}_\omega u(t,\mathrm{e}^{\mathrm{i}x}), & t \geq 0,\ \mathrm{e}^{\mathrm{i}x} \in \mathbb{T}, \\ f(0,\mathrm{e}^{\mathrm{i}x}) = f_0(\mathrm{e}^{\mathrm{i}x}), & \mathrm{e}^{\mathrm{i}x} \in \mathbb{T},\ f_0 \in L^2(\mathbb{T}). \end{cases} \tag{2}$$

As an intermediary step, we also consider the following "H^2 control system", where $P_+f(\mathrm{e}^{\mathrm{i}x}) = \sum_{n\geq 0} \widehat{f}(n)\mathrm{e}^{inx}$ is the Riesz projector and $H^2 = P_+L^2(\mathbb{T})$:

$$\begin{cases} (\partial_t + |D|)f(t,\mathrm{e}^{\mathrm{i}x}) = P_+\mathbb{1}_\omega u(t,\mathrm{e}^{\mathrm{i}x}), & t \geq 0,\ \mathrm{e}^{\mathrm{i}x} \in \mathbb{T}, \\ f(0,\mathrm{e}^{\mathrm{i}x}) = f_0(\mathrm{e}^{\mathrm{i}x}), & \mathrm{e}^{\mathrm{i}x} \in \mathbb{T},\ f_0 \in H^2. \end{cases} \tag{3}$$

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These are control systems with state $f(t,\cdot) \in L^2(\mathbb{T})$ (respectively H^2) and control $u \in L^2([0,T]\times\omega)$. We are interested in which initial states can be steered to 0, namely:

Definition 1.1. We say that $f_0 \in L^2(\mathbb{T})$ (resp. in H^2) is null-controllable (or steerable to 0) from ω in time T in L^2 (resp. in H^2) if there exists $u \in L^2([0,T]\times\omega)$ such that the solution f of the L^2 control system (2) (resp. the H^2 control system (3)) with initial condition f_0 is such that $f(T,\cdot) = 0$.

We denote by $\mathcal{NC}_{L^2}(\omega, T)$ (resp. $\mathcal{NC}_{H^2}(\omega, T)$) the set of initial conditions that are steerable to 0 in L^2 (resp. H^2) from ω in time T.

Using Fourier series, we can check that the operator $|D|$ with domain $W^{1,2}(\mathbb{T})$ is self-adjoint maximal monotone on $L^2(\mathbb{T})$, hence it generates an analytic semigroup $S(t)$ on $L^2(\mathbb{T})$. If $f_0(x) = \sum_{n\in\mathbb{Z}} \widehat{f_0}(n)\mathrm{e}^{\mathrm{i}nx}$, this semigroup is given by

$$S(t)f_0(\mathrm{e}^{\mathrm{i}x}) = \sum_{n\in\mathbb{Z}} \widehat{f_0}(n)\mathrm{e}^{-|n|t}\mathrm{e}^{\mathrm{i}nx}. \tag{4}$$

In other words, the Cauchy problems (2) and (3) are well-posed.

1.2 Overview of the results

The starting observation in our study is that setting $f_{\mathrm{h}}(z) \coloneqq \sum_{n\geq 0} \widehat{f_0}(n)z^n$ and $f_{\mathrm{a}}(z) \coloneqq \sum_{n>0} \widehat{f_0}(-n)\overline{z}^n$ for $|z| < 1$, the semigroup (4) is given by $S(t)f_0(\mathrm{e}^{\mathrm{i}x}) = f_{\mathrm{h}}(\mathrm{e}^{-t+\mathrm{i}x}) + f_{\mathrm{a}}(\mathrm{e}^{-t+\mathrm{i}x})$, where f_{h} is holomorphic and f_{a} is anti-holomorphic.

This suggests that complex and harmonic analysis tools can be used to study these control systems (2)–(3). A central role in our analysis will be played by Bergman spaces. We recall that for a domain $\Omega \subset \mathbb{C}$, the Bergman space $A^2(\Omega)$ is the set of functions f holomorphic on Ω such that $\|f\|^2_{A^2(\Omega)} = \int_\Omega |f|^2 \,\mathrm{d}A < +\infty$, where A is the planar Lebesgue measure (weighted versions are defined integrating against the corresponding weight). We refer to, e.g., [11, §18.1] for an elementary introduction to Bergman spaces.

We start by discussing the null-controllable states for the H^2-control system (3), where free solutions can be written as $f(\mathrm{e}^{-t+\mathrm{i}x})$ for some holomorphic function f.

Combining the above observation and the usual equivalence between null-controllability and the so-called observability inequality [12, Theorem 2.44, 48, Theorem 11.2.1], we prove in section 2.1 the following characterization of null-controllable states:

Proposition 1.2. *Let ω be an open subset of $\mathbb{T}$ and $T > 0$. Let $\Omega_T \coloneqq \{z \in \mathbb{C},\ 1 < |z| < \mathrm{e}^T,\ z/|z| \in \omega\}$ (see fig. 1). Let $f_0 \in H^2$. The following assertions are equivalent:*

- *$f_0 \in \mathcal{NC}_{H^2}(\omega, T)$,*
- *there exists $C_{f_0,T} > 0$ such that for every polynomial $p \in \mathbb{C}[X]$,*

$$|\langle f_0, p\rangle_{H^2}| \leq C_{f_0,T}\|p\|_{A^2(\Omega_T)}. \tag{5}$$

If these assertions are satisfied, we can choose the control u that steers f_0 to 0 such that $\|u\|_{L^2([0,T]\times\omega)} \leq \mathrm{e}^T C_{f_0,T}$, where $C_{f_0,T}$ is the constant that appears in the second assertion.

Our first main result states that when ω is an interval, $\mathcal{NC}_{H^2}(\omega, T)$ does not depend on T:

Theorem 1.3. *Let ω be a non-trivial open interval of $\mathbb{T}$. For every $T, T' > 0$, $\mathcal{NC}_{H^2}(\omega, T) = \mathcal{NC}_{H^2}(\omega, T')$.*

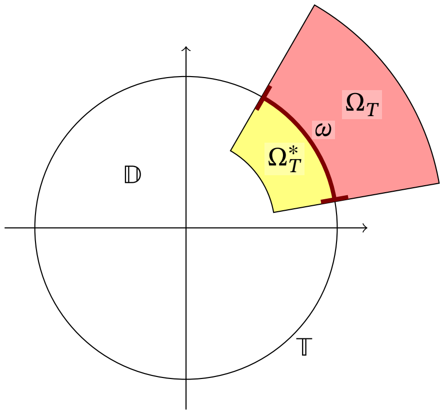

Figure 1: Illustration of Ω_T (red partial ring outside $\mathbb{D}$), and, for later purposes, $\Omega_T^* = \{1/\overline{z},\ z \in \Omega_T\}$ (yellow partial ring inside $\mathbb{D}$).

The proof of theorem 1.3 relies on the theorem of *separation of singularities* which states that under some geometrical assumptions, $A^2(\Omega_1 \cap \Omega_2) = A^2(\Omega_1) + A^2(\Omega_2)$ (see theorem 2.4). In view of theorem 1.3, from now on, we will omit the T in $\mathcal{NC}_{H^2}(\omega, T)$ and write $\mathcal{NC}_{H^2}(\omega)$.

Like the previous theorem, most of our theorems are stated when ω is an interval, but it should be clear while reading the proofs that they still hold when ω is a finite union of intervals that are at positive distance from each other. More pathological ω would require other tools.

We will actually prove a more precise result in theorem 2.2 which gives an estimate of the cost of the control: given $f \in \mathcal{NC}_{H^2}(\omega)$, there is C such that the function u steering the system to zero in time T satisfies

$$\|u\|_{L^2([0,T]\times\omega)} \leq C\max(1, T^{-3/2}).$$

According to our definition of $H^2 = P_+L^2(\mathbb{T})$, functions in H^2 have the form $f\colon \mathrm{e}^{\mathrm{i}x} \in \mathbb{T} \mapsto \sum_{n\geq 0} \widehat{f}(n)\mathrm{e}^{\mathrm{i}nx}$. Since only non negative n are present here, we can actually define $f(z) = \sum_{n\geq 0} \widehat{f}(n)z^n$ for $z \in \mathbb{D} := \{z \in \mathbb{C},\ |z| < 1\}$. This way, we recover the usual definition of the Hardy space H^2 (see, e.g., [45, Definition 17.7 & Theorem 17.12]), with the usual identification of $f \in H^2$ (seen as holomorphic in $\mathbb{D}$) and of its radial limits $f(\mathrm{e}^{\mathrm{i}\theta}) := \lim_{r\to 1^-} f(r\mathrm{e}^{\mathrm{i}\theta})$ [45, Theorem 17.11]. We will make this identification throughout the article.

If $E \subset \mathbb{C}$, we denote the closure of E by E^{cl}. Our main necessary condition for a function $f_0 \in H^2$ to be null-controllable is the following:

Theorem 1.4. *We denote the Riemann sphere by $\widehat{\mathbb{C}}$. Let ω be a non-trivial open interval of $\mathbb{T}$. If $f_0 \in \mathcal{NC}_{H^2}(\omega)$, then f_0 can be extended holomorphically to $\widehat{\mathbb{C}} \setminus \omega^{\mathrm{cl}}$. Denoting this extension also by f_0, we have*

1. $f_0(\infty) = 0$,
2. $\int_{|z|>1} |f_0'(z)|^2 \,\mathrm{d}A(z) < +\infty$.

The proof of theorem 1.4 mostly consists in testing the observability inequality of theorem 1.2 on the *Hardy reproducing kernel* (see definition below). We will call the space of holomorphic functions on $\mathbb{D}_\mathrm{e} := \{z \in \widehat{\mathbb{C}},\ |z| > 1\}$ satisfying 1. and 2. the Dirichlet space of the extended outer disk and denote it by $\mathcal{D}(\mathbb{D}_\mathrm{e})$. Note that by a computation in polar coordinates, for $h(z) = \sum_{n>0} \widehat{h}(-n)z^{-n}$ holomorphic on $\mathbb{D}_\mathrm{e}$, $h' \in L^2(\mathbb{D}_\mathrm{e})$ if and only if $\sum_{n<0} |n||\widehat{h}(n)|^2 < +\infty$. Using a natural connection with holomorphic functions on $\widehat{\mathbb{C}} \setminus \mathbb{T}$ [10, Theorem 5.3.1], we will see that every $f_0 \in \mathcal{NC}_{H^2}(\omega)$ can be written as the so-called *Cauchy-transform* of a function supported on ω. More precisely given

$g \in L^2(\omega)$, we can write:

$$\int_\omega \frac{g(w)}{2\pi(1-\overline{w}z)}|\mathrm{d}w| = \begin{cases} P_+g(z) = \sum_{n\geq 0}\widehat{g}(n)z^n, & |z|<1, \\ -P_-g(z) = -\sum_{n>0}\dfrac{\widehat{g}(-n)}{z^n}, & |z|>1, \end{cases} \tag{6}$$

and the left-hand side is called the *Cauchy transform of* g (here $P_- = I - P_+$ with its natural extension to $\mathbb{D}_\mathrm{e}$). With this, we are able to rephrase theorem 1.4 as:

Theorem 1.5. *Let ω be a non-trivial open interval of $\mathbb{T}$. Let $f_0 \in \mathcal{NC}_{H^2}(\omega)$. Then there exists $g \in L^2(\mathbb{T})$, such that*

- $\operatorname{supp} g \subset \omega^{\mathrm{cl}}$,
- $\sum\limits_{n<0} |n||\widehat{g}(n)|^2 < +\infty$,
- $f_0 = P_+g$.

Another consequence of theorem 1.4 is that we cannot steer non zero analytic initial states to zero.

Corollary 1.6. *Let ω be a non-trivial open interval of $\mathbb{T}$. If $f_0 \in H^2$ extends holomorphically to a disk larger than $\mathbb{D}$ and $f_0 \in \mathcal{NC}_{H^2}(\omega)$, then $f_0 = 0$.*

We also get a sufficient condition for an initial condition to be null-controllable:

Theorem 1.7. *Let ω be a non-trivial interval of $\mathbb{T}$ with $\partial\omega = \{\zeta_1, \zeta_2\}$. Suppose $g \in L^2(\mathbb{T})$ with*

- $\operatorname{supp} g \subset \omega^{\mathrm{cl}}$,
- $\sum\limits_{n<0} |n||\widehat{g}(n)|^2 < +\infty$,
- $\int_\mathbb{T} \frac{|g(\zeta)|^2}{|(\zeta-\zeta_1)(\zeta-\zeta_2)|}|\mathrm{d}\zeta| < \infty$.

Then $P_+g \in \mathcal{NC}_{H^2}(\omega)$.

The proof relies on the notion of *Carleson measures* and *pseudo-Carleson measures* for Bergman spaces defined as follows. We consider $\Omega \subset \mathbb{C}$ open bounded simply connected. We say that a positive measure μ on Ω is a Carleson measure if $\left(\int_\Omega |f(z)|^2 \,\mathrm{d}\mu(z)\right)^{1/2} \lesssim \|f\|_{A^2(\Omega)}$ for $f \in A^2(\Omega)$. We say that a (possibly complex) measure μ on Ω is a pseudo-Carleson measure if $\left|\int_\Omega f(z)^2 \,\mathrm{d}\mu(z)\right|^{1/2} \lesssim \|f\|_{A^2(\Omega)}$ for $f \in A^2(\Omega)$. Of course, a Carleson measure is also a pseudo-Carleson measure. One important theorem is the characterization of Carleson measure: μ is a Carleson measure if and only if $\mu(B(z,r)) \lesssim r^2$ for every ball $B(z,r)$ whose radius r is comparable to $\mathrm{d}(z,\partial\Omega)$ [22, Theorem 2]. We refer to Gonzalez' article [22] for further discussion of Carleson measures and to Xiao's article [49] for a discussion of pseudo-Carleson measure when $\Omega = \mathbb{D}$.

As a matter of fact, Stokes' formula will give us a necessary and sufficient condition expressed in terms of pseudo-Carleson measures (theorem 2.5). Using the characterization of Carleson measures, we will see that the integrability hypothesis of theorem 1.7 indeed implies our pseudo-Carleson measure condition of theorem 2.5.

In the following, $W^{k,p}$ are the usual Sobolev spaces, see, e.g., [33, Definition 7.14] for $k \in \mathbb{N}$ and [32, Definition 1.12] for $0 < k < 1$. Note that according to theorem 1.6, the complementary set of $\mathcal{NC}_{H^2}(\omega)$ contains every nonzero polynomial, hence it is dense in every $H^2 \cap W^{k,2}(\mathbb{T})$. We can also deduce from theorem 1.7 that $\mathcal{NC}_{H^2}(\omega)$ is dense in every $H^2 \cap W^{k,2}(\mathbb{T})$.

Corollary 1.8. *Let ω be a non-empty open interval of $\mathbb{T}$. For every $k \in \mathbb{N}$, the set $\mathcal{NC}_{H^2}(\omega) \cap W^{k,2}(\mathbb{T})$ is dense in $H^2 \cap W^{k,2}(\mathbb{T})$.*

The proof of theorem 1.8 is in section 2.3. Note that $W^{k,2}(\mathbb{T}) \cap H^2$ is a closed subspace of $W^{k,2}(\mathbb{T})$, and we endow $W^{k,2}(\mathbb{T}) \cap H^2$ with the $W^{k,2}(\mathbb{T})$ scalar product.

There is still a gap between the necessary condition of theorem 1.5 and the sufficient condition of theorem 1.7. However, if we assume a priori regularity on f_0, we get a necessary and sufficient condition:

Corollary 1.9. *Let ω be a non-trivial open interval of $\mathbb{T}$. Let $f_0 \in H^2 \cap W^{1/2,2}(\mathbb{T})$. Then $f_0 \in \mathcal{NC}_{H^2}(\omega)$ if and only if there exists $g \in L^2(\mathbb{T})$ such that*

- $\operatorname{supp} g \subset \omega^{\mathrm{cl}}$,
- $g \in W^{1/2,2}(\mathbb{T})$,
- $f_0 = P_+ g$.

Observe that the first two conditions on g in this corollary can be synthesized as $g \in W^{1/2,2}_{00}(\omega)$. Indeed, $W^{1/2,2}_{00}(\omega)$ is by definition the set of functions in $L^2(\omega)$ such that their extension by 0 on $\mathbb{T}$ is in $W^{1/2,2}(\mathbb{T})$ [32, Definition 1.85]. The proof of theorem 1.9 mostly consists in plugging theorems 1.5 and 1.7 with some Hardy-type inequality satisfied by functions in $W^{1/2,2}_{00}$ [32, Theorem 1.87].

We can also get another equivalent condition to $f_0 \in \mathcal{NC}_{H^2}(\omega)$ although less tractable than the corollary above (it involves harmonic extension to Ω_T). Using Stokes formula, we will see in section 2.4 that the following condition seems straightforward (but the rigorous proof takes quite the roundabout way):

Theorem 1.10. *Let ω be a non-trivial open interval of $\mathbb{T}$. Let $g \in L^2(\omega)$, and let u be the solution of $\Delta u = 0$ on Ω_T, $u(z) = zg(z)$ on ω, $u = 0$ on $\partial\Omega_T \setminus \omega$ (see section B and theorem B.3 for the definition). The following assertions are equivalent:*

- $P_+ g \in \mathcal{NC}_{H^2}(\omega)$,
- $\partial_z u \in L^2(\Omega_T)$.

The study of the H^2 control system (3) is the stepping stone to the study of the full L^2 control system (2). We are able to prove in section 3 the following link between $\mathcal{NC}_{H^2}(\omega)$ and $\mathcal{NC}_{L^2}(\omega, T)$:

Theorem 1.11. *Let ω be a strict open interval of $\mathbb{T}$ and let $T > 0$. Then, for every $f_0 \in L^2(\mathbb{T})$.*

$$f_0 \in \mathcal{NC}_{L^2}(\omega, T) \Leftrightarrow \begin{cases} P_+ f_0 \in \mathcal{NC}_{H^2}(\omega), \\ P_+ \overline{f_0} \in \mathcal{NC}_{H^2}(\omega). \end{cases}$$

Recall that, as explained at the beginning of this section, free solutions of the half-heat equation are sums of holomorphic and anti-holomorphic functions in $\mathrm{e}^{-t+\mathrm{i}x}$. If we are to translate the results on $\mathcal{NC}_{H^2}(\omega)$ into results on $\mathcal{NC}_{L^2}(\omega, T)$, we need a way to properly "separate" the dynamics of the positive and negative frequencies just by looking at the control region ω. Fortunately, an inequality by Friedrichs [20] turns out to be just the right tool, and is the main technical ingredient of the proof of theorem 1.11.

Theorem 1.11 allows us to study the set $\mathcal{NC}_{L^2}(\omega, T)$ by using the results on $\mathcal{NC}_{H^2}(\omega, T)$. In particular, since $\mathcal{NC}_{H^2}(\omega, T)$ does not depend on T (theorem 1.3), the right-hand side in the previous equivalence does not depend on T. Hence:

Corollary 1.12. *Let ω be a non-trivial open interval of $\mathbb{T}$. The set $\mathcal{NC}_{L^2}(\omega, T)$ does not depend on T: for all $T', T > 0$, $\mathcal{NC}_{L^2}(\omega, T) = \mathcal{NC}_{L^2}(\omega, T')$.*

We will also prove the following analog of theorem 1.6.

Corollary 1.13. *Let ω be a strict interval of $\mathbb{T}$ and let $T > 0$. If $f_0 \in L^2(\mathbb{T})$ is analytic on $\mathbb{T}$ and $f_0 \in \mathcal{NC}_{L^2}(\omega, T)$, then $f_0 = 0$.*

Theorem 1.9 does not transfer so nicely to the L^2 control system (we would find that $\mathcal{NC}_{L^2}(\omega) \cap W^{1/2,2}(\mathbb{T}) = \{P_+g_+ + (1-P_+)g_-,\ g_\pm \in W^{1/2,2}_{00}(\omega),\ \int_{\mathbb{T}} g_+ = \int_{\mathbb{T}} g_-\}$; we leave the proof to the interested reader), but we can still prove that the space of null-controllable initial data is dense:

Corollary 1.14. *For every $k \in \mathbb{N}$, the set $\mathcal{NC}_{L^2}(\omega) \cap W^{k,2}(\mathbb{T})$ is dense in $W^{k,2}(\mathbb{T})$.*

Finally, our method gives some results on the half-heat equation on a segment (see appendix C for some results with Dirichlet boundary conditions) and on the reachable space for the half-heat equation (see appendix D)

1.3 Discussion and examples

Let us first mention that while the half-heat equation is not null-controllable (i.e., $\mathcal{NC}_{L^2}(\omega) \neq L^2(\mathbb{T})$, as seen for instance in theorem 1.6), it is *approximately controllable*, meaning that for every $f_0, f_1 \in H^2$ (resp. in $L^2(\mathbb{T})$), for every $\epsilon > 0$, there exists $u \in L^2([0,T]\times\omega)$ such that the solution of the control system (3) (resp. (2)) with $f(0,\cdot) = f_0$ is such that $\|f(T,\cdot) - f_1\|_{L^2} < \epsilon$. Indeed, this approximate controllability is equivalent to *unique continuation* [12, Theorem 2.43]. This is standard, but let us quickly prove that unique continuation holds for the L^2-control system (2). Let $g_0 \in L^2(\mathbb{T})$ and assume that $g(t, e^{ix}) = S(t)^* g_0(e^{ix}) = \sum_{n\in\mathbb{Z}} \widehat{g_0}(n) e^{-|n|t + inx}$ vanishes on $[0,T]\times\omega$. We see that g is real-analytic in t and in e^{ix} on $\mathbb{R}_+^* \times \mathbb{T}$, and since it vanishes on $[0,T]\times\omega$, it vanishes everywhere, proving the claim.

Let us now give some examples of functions that are or are not null-controllable.

Our necessary condition theorem 1.5 is efficient at proving that an initial state is not null-controllable: anything that is not smooth on $\mathbb{T}\setminus\omega$ is not null-controllable, but anything that is *too smooth* (analytic) on the whole space is not null-controllable neither (theorem 1.6).

Giving a null-controllable initial state with a closed form expression is harder. For some $0 < a < \pi$, consider the case $\omega = \{|z| = 1,\ |\arg(z)| < a\}$. For $0 < \theta_0 \leq a$, consider g a "triangle" function

$$g(e^{i\theta}) = \begin{cases} 0, & -\pi < \theta < -\theta_0, \\ \theta_0 + \theta, & -\theta_0 < \theta < 0, \\ \theta_0 - \theta, & 0 < \theta < \theta_0, \\ 0, & \theta_0 < \theta < \pi. \end{cases}$$

Then straightforward (but tedious) computations show that

$$P_+g(z) = \theta_0^2 - 2\,\mathrm{Li}_2(z) + \mathrm{Li}_2(e^{i\theta_0}z) + \mathrm{Li}_2(e^{-i\theta_0}z),$$

where Li_2 is the dilogarithm defined by $\mathrm{Li}_2(z) = \sum_{n>0} \frac{z^n}{n^2}$. We plot P_+g in fig. 2. Since $g \in W^{1,2}_0(\omega)$, $P_+g \in \mathcal{NC}_{H^2}(\omega)$. This is the simplest null-controllable initial state with a closed-form expression we could find.

A difficult question is to decide in general whether a given function $f_0 \in H^2$ is the Riesz projection of a function $g \in L^2(\mathbb{T})$ with $\mathrm{supp}(g) \subset \omega^{\mathrm{cl}}$, and moreover extending holomorphically to $\widehat{\mathbb{C}} \setminus \omega^{\mathrm{cl}}$. This question is related to the delicate problem of describing ranges of so-called Toeplitz operators.

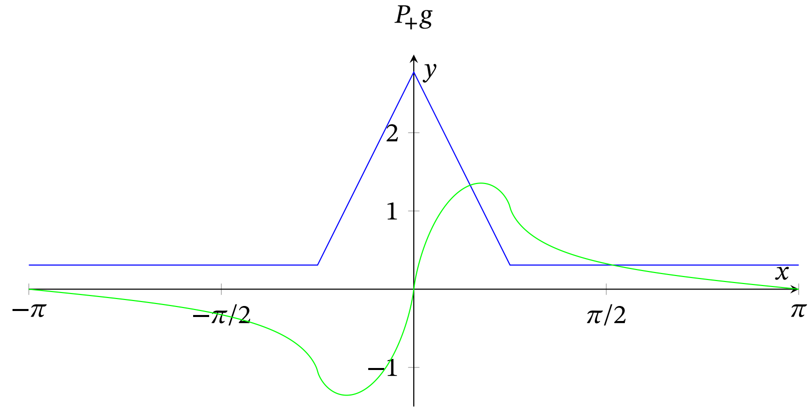

Figure 2: A plot of $\theta \mapsto P_+g(e^{i\theta})$ for $\theta_0 = \pi/4$. The real part is in blue, the imaginary part in green.

Still, it is clear that if f_0 is the Riesz projection of $g \in L^2(\mathbb{T})$ and $\operatorname{supp}(g) \subset \omega^{\mathrm{cl}}$, then g is unique. Indeed, if $P_+g = 0$, then $P_-g = 0$ on $\mathbb{T} \setminus \omega^{\mathrm{cl}}$ which by standard uniqueness results in H^2 [45, Theorem 17.18] (or $\overline{H^2}$) implies that $g = 0$.

In the language of inverse problem, the problem "recover $g \in L^2(\omega)$ from P_+g" is identifiable. However, we do not have an inequality $\|g\|_{L^2(\omega)} \lesssim \|P_+g\|_{L^2}$: consider $\chi \in C_c^\infty(\omega)$ and $\chi_k(x) = e^{-ikx}\chi(x)$. Then $\widehat{\chi_k}(n) = \widehat{\chi}(n+k)$, hence $\|P_+\chi_k\|_{L^2}^2 = 2\pi \sum_{n\geq k} |\widehat{\chi}(n)|^2 \to 0$ as $k \to +\infty$ while $\|\chi_k\|_{L^2(\omega)} = c > 0$. In the language of inverse problems, the problem above is not stable.

We also mention Tumarkin's theorem [10, Theorem 5.3.1], which we actually use to deduce theorem 1.5 from theorem 1.4. So probably Tumarkin's theorem would not tell us more than theorem 1.4. We refer to [10, Chapter 5] for related results.

Let us finally discuss how the spaces (for the function g) that appear in theorems 1.5, 1.7, 1.9 and 1.10 compare to each other. It will be convenient to have a notation for the space of functions with different type of regularity/integrability for positive and negative frequencies.

Definition 1.15. We assume that ω is a strict interval of $\mathbb{T}$ with endpoints ζ_1 and ζ_2. Given $s, t \in \mathbb{R}$, we set

$$X_{t-}^{s+}(\omega) := \{g \in L^2(\mathbb{T}),\ \operatorname{supp}(g) \subset \omega^{\mathrm{cl}},\ P_+g \in W^{s,2}(\mathbb{T}),\ (I - P_+)g \in W^{t,2}(\mathbb{T})\},$$

$${}^0X_{t-}^{s+}(\omega) := X_{t-}^{s+}(\omega) \cap \left\{ \int_{\mathbb{T}} \frac{|g(\zeta)|^2}{|(\zeta-\zeta_1)(\zeta-\zeta_2)|} |d\zeta| < \infty \right\},$$

$$\widetilde{X}(\omega) := \{g \in L^2(\omega),\ \partial_z P^{\Omega_T} g \in L^2(\Omega_T)\},$$

where $P^{\Omega_T} : L^2(\partial\Omega_T) \to h(\Omega_T)$ is the Poisson integral operator for Ω_T. If $g \in L^2(\omega)$ and $f_0 = P_+g$, with these notations, the previous theorems can be rephrased as

$$f_0 \in \mathcal{NC}_{H^2}(\omega) \cap W^{1/2,2}(\mathbb{T}) \Leftrightarrow g \in X_{1/2-}^{1/2+}(\omega), \qquad \text{(theorem 1.9)}$$

$$f_0 \in \mathcal{NC}_{H^2}(\omega) \implies g \in X_{1/2-}^{0+}(\omega), \qquad \text{(theorem 1.5)}$$

$$f_0 \in \mathcal{NC}_{H^2}(\omega) \Leftrightarrow g \in \widetilde{X}(\omega), \qquad \text{(theorem 1.10)}$$

$$g \in {}^0X_{1/2-}^{0+}(\omega) \implies f_0 \in \mathcal{NC}_{H^2}(\omega). \qquad \text{(theorem 1.7)}$$

Thus, we have

$$X_{1/2-}^{1/2+}(\omega) \subseteq {}^0X_{1/2-}^{0+}(\omega) \subseteq \widetilde{X}(\omega) \subseteq X_{1/2-}^{0+}(\omega).$$

See, e.g., [32, Theorem 1.87] for the first inclusion. The space $X^{1/2+}_{1/2-}(\omega) = W^{1/2,2}_{00}(\omega)$ that appears in theorem 1.9 is the simplest, or maybe the least unusual of the bunch. The triangle function g above belongs in this space (and even in $W^{1,2}_0(\omega)$).

We can construct a function in $X^{0+}_{1/2-}(\omega) \setminus X^{1/2+}_{1/2-}(\omega)$ the following way: assuming $1 \in \omega$, first consider $h(z) = -\sum_{n>0} z^n/n = \log(1-z)$. Then $P_+h \notin W^{1/2,2}(\mathbb{T})$ and $(I-P_+)h = 0 \in W^{1/2,2}(\mathbb{T})$. Consider also $\chi \in C^\infty_c(\omega)$ such that $\chi = 1$ on a neighborhood of 1. Finally, set $g = \chi h$. Then, we have $g \in L^2(\omega)$, and writing $\widehat{g}(n) = \sum_{k<n} \widehat{\chi}(k)/(n-k)$, we see that $\widehat{g}(n)$ decays faster than any polynomial as $n \to -\infty$. In particular, $(I - P_+)g \in W^{1/2,2}(\mathbb{T})$. On the other hand, $g \notin W^{1/2,2}(\mathbb{T})$, as that would imply $h = (1-\chi)h + g \in W^{1/2,2}(\mathbb{T})$, a contradiction. Thus g is a function that satisfies the necessary condition of theorem 1.5 that is not in $W^{1/2,2}(\mathbb{T})$. Since the support of g is compact in ω, g also satisfies the sufficient condition of theorem 1.7.

Thus, in the inclusion chain above, we actually have $X^{1/2+}_{1/2-}(\omega) \subsetneq {}^0X^{0+}_{1/2-}(\omega)$.

Next, assume that $\omega = \{e^{it},\ 0 < t < a\}$ for some $0 < a < 2\pi$, choose $\chi \in C^\infty_c([0,a))$ with $\chi = 1$ on $[0,\epsilon]$ for some $0 < \epsilon < a$. Finally, set $g(e^{it}) = \chi(t)\mathbb{1}_{[0,a)}(t)e^{-i/t}$. We can estimate $\widehat{g}(n)$ using some non-stationary phase (for $n < 0$) and stationary phase (for $n > 0$) arguments (see [51, Lemma 3.10 and Theorem 3.11] for the general idea) to get:

$$\forall N > 0,\ \widehat{g}(n) = O_{n\to-\infty}(n^{-N}), \qquad |\widehat{g}(n)| \sim_{n\to+\infty} cn^{-3/4}.$$

In particular, $g \in X^{0+}_{1/2-}(\omega)$ but does not satisfy the integrability condition of ${}^0X^{0+}_{1/2-}(\omega)$. Hence, ${}^0X^{0+}_{1/2-}(\omega) \subsetneq X^{0+}_{1/2-}(\omega)$. We could not prove whether the other inclusions are strict or not.

1.4 Bibliographical comments

Control of the fractional heat equation The controllability of the fractional heat equation

$$(\partial_t + (-\Delta)^\alpha u)f = Bu$$

has been studied since 2006 by Miller [40] and Micu and Zuazua [37].

More precisely, Miller [40] shows that the Lebeau-Robbiano control strategy (used to prove the null-controllability of the heat equation [31]) is also adapted to prove the null-controllability of the fractional heat equation when $\alpha > 1/2$.

On their end, Micu and Zuazua use the moment method (to prove the null-controllability of the heat equation in dimension one [19]). They also remark that this strategy works for the fractional heat equation when $\alpha > 1/2$.

In the case $\alpha \le 1/2$, Miller, Micu and Zuazua (see also [17, Appendix]) consider *shaped controls*, i.e., the right-hand side of the equation is $u(t)h(x)$ (instead of $\mathbb{1}_\omega u(t,x)$ as we are considering in this article). Using Müntz' theorem, they show that the fractional heat equation with $\alpha \le 1/2$ is not null-controllable: for every control shape $h(x)$ and $T > 0$ there exists an initial state that cannot be steered to 0 in time T.

We emphasize that the kind of control they consider (shaped control) is different to the controls we are considering (internal control). On the surface, both kinds of control seem to give similar results in dimension one: if $\alpha > 1/2$ the fractional heat equation is null-controllable with both, and if $\alpha \le 1/2$, the fractional heat equation is not null-controllable, whatever kind of control (see [29, 2, 35] and [30, Theorem 4] for lack of null-controllability with internal controls).

However, differences in controllability properties appear when we look more closely. Let us denote $\mathcal{NC}_h(T)$ the space of null-controllable initial states for the half-heat equation with shaped control with shape h (Equation 1.1 with $f = h$ in Micu and Zuazua'a article [37]). Then, for some control shapes, $\mathcal{NC}_h(T) = \{0\}$ [37, Theorem 3.1] (but if the control shape is analytical and non-zero,

$\mathcal{NC}_h(T) \neq \{0\}$ [37, Proposition 3.3]). On the other end, with internal controls, our results show that the space of null-controllable states $\mathcal{NC}_{L^2}(\omega)$ is somewhat large (theorems 1.7 to 1.9). Let us also mention that theorem 1.6 was previously proved by the second author with a different proof [28, Théorème 2.5].

These differences of controllability properties indicate that our methods do not adapt for shaped control. Indeed, an important point of our analysis consists in identifying the right-hand side of the observability inequality and a Bergman space norm (theorem 1.2), but nothing of this sort is possible for shaped controls (see [37, eq. (5.1)] for the observability inequality associated to shaped controls).

The case $\alpha = 1/2$ (i.e., the half-heat equation) is interesting for another reason: in a sense, it can be embedded in the *Baouendi-Grushin heat equation* (where $\omega \subset \Omega \coloneqq (-1,1)\times\mathbb{T}$),

$$\begin{cases}(\partial_t - \partial_y^2 - y^2\partial_x^2)f(t,y,x) = \mathbf{1}_\omega u(t,y,x), & (t,y,x)\in\mathbb{R}_+\times(-1,1)\times\mathbb{T},\\ f(0,y,x) = f_0(y,x), & (y,x)\in(-1,1)\times\mathbb{T}.\end{cases}$$

This time it is controlled from the interior of the domain (we have interchanged x and y here in order to keep $x\in\mathbb{T}$ to be consistant with the notation of the present article). Consider now $(v_{n,k})$, the eigenfunctions of $-\partial_y^2 + (ny)^2$ (with Dirichlet boundary conditions) associated to the eigenvalues $\lambda_{n,k}$, then $\Phi_{n,k}(y,x) = v_{n,k}(y)e^{inx}$ are eigenfunctions of $-\partial_y^2 - y^2\partial_x^2$ and free solution can be expressed as $f(t,y,x) = \sum_{n,k} \mathrm{e}^{-t\lambda_{n,k}} c_{n,k}\Phi_{n,k}(y,x)$. Since $\lambda_{n,0} = n + o(n)$ [15, Theorem 4.23], it is natural to introduce the corresponding differential operator on $\mathbb{T}$ by

$$|D_x|\Big(\sum_n a_n e^{inx}\Big) = \sum_n a_n|n|\mathrm{e}^{inx}.$$

With this in mind, after having established that the so-called half-heat equation $(\partial_t + |D_x|)f(t,x) = \mathbb{1}_\omega u(t,x)$ (f being in a suitable subspace of H^2) is not null-controllable on ω in any time, the second author deduces that the Grushin equation is not null controllable in any time [30, Theorem 4].

This is the starting point of this paper. Indeed, knowing that the half-heat equation is not null-controllable in any time raises the question which states can be steered to zero.

Complex analysis tools for control theory Complex analysis tools have proven to be powerful in the study of several control problems, and our article is in the line of this tradition.

In particular, holomorphic function spaces and analytic function theory have shown to be efficient in describing reachable states in certain control problems. Let us mention the classical heat equation. For an interval $I\subset\mathbb{R}$, this is given by

$$\begin{cases}\partial_t y(t,x) - \partial_x^2 y(t,x) = 0, & t>0,\ x\in I,\\ y(0,x) = f(x), & x\in I,\end{cases}$$

Aikawa, Hayashi and Saitoh [1] were considering this problem when $I = \mathbb{R}_+$ and the equation is controlled in $x_0 = 0$ by an L^2-function in some finite time. As it turns out, the solution $y(T,\cdot)$ ($T>0$ fixed) can be expressed as a certain Laplace-type transform for which the range is characterized as a weighted Bergman space on the sector $\{z = x+iy\in\mathbb{C} : x>0, |y|<x\}$.

Another prominent situation is when I is a finite interval, say $I = (0,\pi)$, and the heat equation is controlled from $x_0 = 0$ and $x_1 = \pi$:

$$\begin{cases}\partial_t y(t,x) - \partial_x^2 y(t,x) = 0, & t>0,\ x\in(0,\pi),\\ y(t,0) = u_0(t),\ \ y(t,\pi) = u_\pi(t), & t>0,\\ y(0,x) = f(x), & x\in(0,\pi),\end{cases}$$

where u_0 and u_π are L^2-functions (Dirichlet control; other types of boundary conditions can be considered). We know thanks to an argument by Seidman [46], that for a well-posed linear control system that is null-controllable in arbitrarily small time, the reachable space does not depend on time. This heat equation is null-controllable in any time, and thus the reachable states do not depend on time. Consider u_0 and u_π on some finite interval $(0, T)$. In this situation, pioneering work by Fattorini and Russell [19] showed that the reachable states extend to holomorphic functions on the square D one diagonal of which is the interval $(0, \pi)$. Subsequent work focused on a more precise description of the reachable space. Martin, Rosier and Rouchon [36] proved that functions holomorphic in a disk with a certain radius are reachable. This was significantly improved by Dardé and Ervedoza [14] who establish that functions holomorphic in a neighborhood of D are reachable. Tuscnak, Hartmann and Kellay [23] then showed that the reachable space is sandwiched between two spaces of holomorphic spaces on D, the Smirnov space, which is a kind of companion space to the Hardy space, and the Bergman space. Using completely different techniques, the next steps were laid down in [41] and [26], where the reachable space was identified as a sum of two Bergman spaces — unweighted in the former paper, weighted in the latter. The final step was reached in [24] where the reachable space was characterized as the Bergman space on D. A central tool of this result was the so-called separation of singularities problem which is a purely complex analysis problem (see below), and which will show to be of relevance also in the problem we will consider here.

Null-controllable states for other equations Most papers on null-controllability either

- prove that a control system is null-controllable, e.g., the heat equation [31, 21], Baouendi-Grushin equation in some context [6], fractional heat with exponent larger than 1/2 [40, 39, 17, 3] and many more,
- or prove it is not null-controllable, e.g., fractional heat equation with exponent at most 1/2 [30, 29, 2] Baouendi-Grushin in other contexts [6, 30] and many more.

In the second case, one could ask which initial conditions can be steered to 0, but this is less studied. We are aware of the following papers that tackle this question, in addition to the previously discussed Micu-Zuazua's paper [37]:

- The heat equation (possibly with some quadratic potential) on the half-line [38, 13], where no non-zero initial state can be steered to 0,
- The Baouendi-Grushin equation controlled on a symmetric vertical band [7, Theorem 1.4], where initial states that are regular enough can be steered to 0 in a smaller time than other initial states,
- The heat equation on $\mathbb{R}$ with "less and less control at ∞" [9], where initial states in a suitably weighted L^2 space can be steered to 0.
- A heat-wave cascade system, where an explicit subspace of null-controllable initial conditions depending on the coupling is identified [34].

Our article is an addition to this short list.

1.5 Notations

We will introduce notations along the way, but to help the reader, we collect here the recurring notations that appear in this article.

General notations	
$A \lesssim B$	$\exists C > 0,\ A \leq CB$ (A, B depend on some parameters)
$A \asymp B$	$A \lesssim B$ and $B \lesssim A$
$\langle f, g\rangle_H$	Scalar product in the Hilbert space H, left-linear and right-antilinear
Sets	
$\widehat{\mathbb{C}}$	Riemann sphere ($\mathbb{C} \cup \{\infty\}$)
$\mathbb{D}$	Unit disk $\{z \in \mathbb{C},\ \lvert z\rvert < 1\}$
$\mathbb{T}$	Complex unit circle $\{z \in \mathbb{C},\ \lvert z\rvert = 1\}$
$\mathbb{D}_{\mathrm{e}}$	Extended exterior of the unit disk $\{z \in \widehat{\mathbb{C}},\ \lvert z\rvert > 1\}$
ω	Control region, non-trivial open connected subset of $\mathbb{T}$
Ω_T	$\{1 < \lvert z\rvert < \mathrm{e}^T,\ \arg(z) \in \omega\}$ (see fig. 1)
Ω_T^*	$\{\mathrm{e}^{-T} < \lvert z\rvert < 1,\ \arg(z) \in \omega\}$ (see fig. 1)
$\overline{E}$	When $E \subset \mathbb{C}$, $\{\overline{z},\ z \in E\}$
E^{cl}	Closure of E
$\mathrm{d}(E, F)$	Distance between two subsets of $\mathbb{C}$
Spaces	
$L^2(\mathbb{T})$	Lebesgue space on $\mathbb{T}$ with respect to linear measure
$L^2(\Omega)$	Lebesgue space on Ω with respect to area measure
$h(\Omega)$	Space of harmonic functions on Ω
$H(\Omega)$	Space of holomorphic functions on Ω
H^2	Hardy space ($H^2 = H^2(\mathbb{D}) = P_+L^2(\mathbb{T})$, see below for P_+)
$A^2(\Omega)$	Bergman space (holomorphic functions in $L^2(\Omega)$)
$\mathcal{D}(\Omega)$	Dirichlet space (holomorphic functions with derivative in $L^2(\Omega)$)
$W^{k,p},\ W_0^{k,p},\ W_{00}^{k,p}$	Sobolev spaces
$\mathcal{NC}_{H^2}(\omega, T),\ \mathcal{NC}_{L^2}(\omega, T)$	Spaces of null-controllable initial states (theorem 1.1)
Functions	
k_u	Hardy reproducing kernel: $\forall f \in H^2$, $f(u) = \langle f, k_u\rangle_{H^2}$. $k_u(z) = \frac{1}{2\pi(1-\overline{u}z)}$
k_u^Ω	Bergman ($A^2(\Omega)$) reproducing kernel: $\forall f \in A^2(\Omega)$, $f(u) = \langle f, k_u^\Omega\rangle_{A^2(\Omega)}$
Operators	
P_+	Riesz projector $L^2(\mathbb{T}) \to L^2(\mathbb{T})$ (projection on nonnegative frequencies)
P_-	$I - P_+ : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ (I identity operator)
$S(t)$	Half-heat semigroup (eq. (4))
P^Ω	Poisson kernel (and associated integral operator $L^2(\partial\Omega) \to h(\Omega)$)
P^{e}	Poisson Kernel on $\mathbb{D}_{\mathrm{e}}$
$\partial_z, \partial_{\overline{z}}$	Wirtinger derivatives $(\partial_x - \mathrm{i}\partial_y)/2$ and $(\partial_x + \mathrm{i}\partial_y)/2$ respectively
$\mathcal{F}_T$	Input-to-output map (theorem 2.1)

Measures

$\lvert \mathrm{d}z \rvert$	Linear (1-dimensional Hausdorff) measure
$\mathrm{d}A$	Area (2-dimensional Lebesgue) measure

2 The H^2 problem

2.1 The observability inequality

Let us start by defining the *input-to-output map*; a usual tool in control theory.

Definition 2.1. Let $T > 0$ and $\omega \subset \mathbb{T}$ measurable. We define the input-to-output map associated to the L^2 control system (2) $\mathcal{F}_T : L^2(0, T;\ L^2(\omega)) \to L^2(\mathbb{T})$ by

$$\forall u \in L^2(0, T;\ L^2(\omega)),\ \mathcal{F}_T u(\mathrm{e}^{\mathrm{i}x}) := \int_0^T S(T-s)(\mathbb{1}_\omega u(s,\cdot))(\mathrm{e}^{\mathrm{i}x})\,\mathrm{d}s.$$

We also define the input-to-output map associated to the H^2 control system (3) $\mathcal{F}_T^+ : L^2(0, T;\ L^2(\omega)) \to H^2$ by

$$\forall u \in L^2(0, T;\ L^2(\omega)),\ \mathcal{F}_T^+ u(\mathrm{e}^{\mathrm{i}x}) := \int_0^T S(T-s)(P_+\mathbb{1}_\omega u(s,\cdot))(\mathrm{e}^{\mathrm{i}x})\,\mathrm{d}s.$$

These definitions correspond to the integral term that appears in Duhamel's formula for the control systems. Notice also that $\mathcal{F}_T^+ = P_+\mathcal{F}_T$.

We now prove theorem 1.2 by combining the change of variables $z = \mathrm{e}^{-t+\mathrm{i}x}$ with the equivalence between null-controllability and observability.

Proof of theorem 1.2. The case $f_0 = 0$ is immediate with $C_{f_0,T} = 0$. We now assume $f_0 \neq 0$.

By definition of $S(T)$, the solution f of the H^2-control system (3) is

$$f(T,\cdot) = S(T)f_0 + \mathcal{F}_T^+ u.$$

Hence, a function f_0 is null controllable at time $T > 0$ if and only if $S(T)f_0 \in \operatorname{Rg}\mathcal{F}_T^+$. Consider the operators $C_2 : f_1 \in \mathbb{C}f_0 \mapsto S(T)f_1 \in H^2$ and $C_3 = \mathcal{F}_T^+ : L^2([0,T]\times\omega) \to H^2$. Then, f_0 is null-controllable at time T if and only if $\operatorname{Range}(C_2) \subset \operatorname{Range}(C_3)$.

According to a standard duality lemma [12, Lemma 2.48], this is equivalent to the existence of $C'_{f_0,T} > 0$ such that

$$\forall g \in H^2,\ \|C_2^* g\|_{H^2} \le C'_{f_0,T}\|C_3^* g\|_{L^2(0,T,L^2(\omega))} = C'_{f_0,T}\|(\mathcal{F}_T^+)^* g\|_{L^2(0,T,L^2(\omega))}, \tag{7}$$

and if this holds, there exists a continuous linear map $C_1 : \mathbb{C}f_0 \to L^2([0,T]\times\omega)$ such that $C_2 = C_3C_1$ and $\|C_1\| \le C'_{f_0,T}$. This relation reads $S(T)f_0 = \mathcal{F}_T^+ C_1 f_0$, that is, $-C_1 f_0$ steers f_0 to 0 in time T. Thus, if the estimate (7) holds, we can choose the control u that steers f_0 to 0 such that $\|u\|_{L^2([0,T]\times\omega)} \le C'_{f_0,T}\|f_0\|_{H^2}$.

To compute the left and right-hand sides, we first see from the definition of the semi-group S (see (4)) that if $g \in H^2$, then $S(t)^* g(\mathrm{e}^{\mathrm{i}x}) = S(t)g(\mathrm{e}^{\mathrm{i}x}) = g(\mathrm{e}^{\mathrm{i}x-t})$.

Let us compute the left-hand side. Set ι_{f_0} the injection map from $\mathbb{C}f_0$ to H^2. Then, $C_2 = S(T)\circ\iota_{f_0}$. Routine computations show that $\iota_{f_0}^*$ is the orthogonal projection onto $\mathbb{C}f_0$, that is

$$\iota_{f_0}^* g = \|f_0\|_{H^2}^{-2}\langle g, f_0\rangle_{H^2} f_0.$$

Thus,

$$C_2^* g = \iota_{f_0}^* \circ S(T)^* g = \iota_{f_0}^* \circ S(T) g = \iota_{f_0}^*(g(\mathrm{e}^{-T}\cdot)) = \|f_0\|_{H^2}^{-2} \langle g(\mathrm{e}^{-T}\cdot), f_0\rangle_{H^2} f_0. \tag{8}$$

Thus the observability inequality (7) is equivalent to

$$\forall g \in H^2,\ |\langle g(\mathrm{e}^{-T}\cdot), f_0\rangle_{H^2}| \leq C'_{f_0,T} \|f_0\|_{H^2} \|(\mathcal{F}_T^+)^* g\|_{L^2(0,T,L^2(\omega))}. \tag{9}$$

Let us now compute the right hand side of eq. (7). First, routine computations [12, Lemma 2.47] show that

$$(\mathcal{F}_T^+)^* g(t) = \mathbb{1}_\omega^* P_+ S(T-t)^* g, \tag{10}$$

where $\mathbb{1}_\omega^* : L^2(\mathbb{T}) \to L^2(\omega)$ is the restriction operator. Thus, the right-hand side of the observability inequality (7) is,

$$\begin{aligned}\|(\mathcal{F}_T^+)^* g\|^2_{L^2(0,T;L^2(\omega))} &= \int_0^T \|(\mathcal{F}_T^+)^* g(t)\|^2_{L^2(\omega)}\,\mathrm{d}t = \int_0^T \int_\omega |g(\mathrm{e}^{\mathrm{i}x-(T-t)})|^2\,\mathrm{d}x\,\mathrm{d}t\\ &= \int_0^T \int_\omega |g(\mathrm{e}^{\mathrm{i}x-t})|^2\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

We make the change of variables $z = \mathrm{e}^{\mathrm{i}x-t}$, that satisfies $\mathrm{d}A(z) = |z|^2\,\mathrm{d}t\,\mathrm{d}x$ ($\mathrm{d}A$ is the area measure on $\mathbb{C} \simeq \mathbb{R}^2$). The image of $[0,T]\times\omega$ by this change of variables is $\Omega_T^* = \{z \in \mathbb{C} :\ \arg(z) \in \omega, \mathrm{e}^{-T} < |z| < 1\}$ (see fig. 1). Hence,

$$\|(\mathcal{F}_T^+)^* g\|^2_{L^2(0,T;L^2(\omega))} = \int_{\Omega_T^*} |g(z)|^2 |z|^{-2}\,\mathrm{d}A(z). \tag{11}$$

Since $1 \leq |z|^{-2} \leq \mathrm{e}^{2T}$, plugging this in the right-hand side of (9) proves that the observability inequality (7) (hence the null-controllability of f_0) is equivalent to

$$|\langle g(\mathrm{e}^{-T}\cdot), f_0\rangle_{H^2}| \leq C''_{f_0,T} \|g\|_{A^2(\Omega_T^*)} \tag{12}$$

for every $g \in H^2$. In addition, if the inequality (12) holds, then, the inequality (9) (and hence (7)) holds with $C'_{f_0,T} \leq \|f_0\|_{H^2}^{-1} C''_{f_0,T}$. In this case, we can choose the control u that steers f_0 to 0 such that $\|u\|_{L^2([0,T]\times\omega)} \leq C'_{f_0,T}\|f_0\|_{H^2} \leq C''_{f_0,T}$.

Since polynomials are dense in H^2 [16, Theorem 3.3], and since $H^2 \subset A^2(\mathbb{D}) \subset A^2(\Omega_T^*)$ with continuous inclusions, we can replace "for every $g \in H^2$" above by "for every $g \in \mathbb{C}[X]$".

Finally, the change of function $\tilde{g}(z) = g(\mathrm{e}^{-T}z)$ proves that the inequality (12) can be written as

$$\forall g \in \mathbb{C}[X],\ |\langle f_0, g\rangle_{H^2}| \leq \mathrm{e}^{-T} C''_{f_0,T} \|g\|_{A^2(\Omega_T)}, \tag{13}$$

where $\Omega_T = \mathrm{e}^T \Omega_T^*$ is indeed the one defined in the statement of theorem 1.2. □

2.2 Independence of null-controllable states with respect to time

We first prove that $\mathcal{NC}_{H^2}(\omega, T)$ does not depend on T when ω is an interval. In fact, we are able to estimate the norm of the control as $T \to 0$:

Theorem 2.2. *Let $T > 0$ and $f_0 \in \mathcal{NC}_{H^2}(\omega, T)$. There exists $C > 0$ such that for every $T' > 0$, there exists $u \in L^2([0,T']\times\omega)$ with*

- $S(T')f_0 + \mathcal{F}_{T'}^+ u = 0,$

- $\|u\|_{L^2([0,T']\times\omega)} \leq C\max(1, T'^{-3/2})$.

The first item says that the solution f of the H^2 control system (3) satisfies $f(T',\cdot) = 0$, i.e., $f_0 \in \mathcal{NC}_{H^2}(\omega, T')$. Hence, theorem 2.2 implies theorem 1.3.

A central ingredient will be a result on separation of singularities:

Theorem 2.3 (Corollary 1.2 [24]). *Let $1 < p < +\infty$. Let Ω_1 and Ω_2 be open sets of $\mathbb{C}$ such that $\emptyset \neq \Omega_1 \cap \Omega_2$ is bounded. If $\mathrm{d}(\Omega_1 \setminus \Omega_2, \Omega_2 \setminus \Omega_1) > 0$, then $A^p(\Omega_1 \cap \Omega_2) = A^p(\Omega_1) + A^p(\Omega_2)$.*

We actually need a little bit more information than this reference. Namely, we want the decomposition $A^2(\Omega_1 \cap \Omega_2) \ni g = g_1 + g_2 \in A^2(\Omega_1) + A^2(\Omega_2)$ to be done in a linear continuous way, with an upper bound on the norm of the resulting operator. All of that is done by looking carefully at the proof of the previous theorem, which we will do now:

Theorem 2.4. *Let $R > 0$. Let $\Omega_1, \Omega_2 \subset B(0,R)$ open such that $\mathrm{d}(\Omega_1 \setminus \Omega_2,\ \Omega_2 \setminus \Omega_1) > 0$. There exists $B_i : A^p(\Omega_1 \cap \Omega_2) \to A^p(\Omega_i)$ ($i = 1,2$) linear continuous such that*

- *$\forall g \in A^p(\Omega_1 \cap \Omega_2)$, $g = B_1 g + B_2 g$,*
- *there exists $C_R > 0$ depending only on R such that for all $g \in A^p(\Omega_1 \cap \Omega_2)$,*

$$\|B_i g\|_{A^p(\Omega_i)} \leq \frac{C_R}{\mathrm{d}(\Omega_1 \setminus \Omega_2,\ \Omega_2 \setminus \Omega_1)}\|g\|_{A^p(\Omega_1\cap\Omega_2)}.$$

Proof. The idea of the proof is essentially the same as in [24]. The starting point is the construction of a smooth partition of unity $\{1-\chi, \chi\}$ subordinate to $\{\Omega_1, \Omega_2\}$: $\chi \in C^\infty(\mathbb{C})$ such that $\chi = 0$ on a neighborhood of $(\Omega_1 \setminus \Omega_2)^{\mathrm{cl}}$ and $\chi = 1$ on a neighborhood of $(\Omega_2 \setminus \Omega_1)^{\mathrm{cl}}$ with $0 \leq \chi \leq 1$. The key observation, which is the novelty here, is that χ additionally satisfies

$$\|\nabla \chi\|_{L^\infty(\Omega_1\cup\Omega_2)} \leq \frac{C}{\mathrm{d}(\Omega_1 \setminus \Omega_2,\ \Omega_2 \setminus \Omega_1)}. \tag{14}$$

The construction is standard, but we detail it in section A for the reader's convenience.

Now, given $g \in A^p(\Omega_1 \cap \Omega_2)$, set $h = g\partial_{\overline{z}}\chi$ (initially defined on $\Omega_1 \cap \Omega_2$ and extended by 0) and $u = \frac{1}{z} * h$, the operators B_i are defined as follows:

$$\begin{aligned} B_1 g &= \chi g - u, \\ B_2 g &= (1-\chi)g + u, \end{aligned} \tag{15}$$

where the functions on the right hand side are extended trivially outside $\Omega_1 \cap \Omega_2$ (these extensions are obviously smooth on Ω_1 and Ω_2 respectively).

By definition $B_1 g + B_2 g = g$. In addition, since $1/z$ is the fundamental solution for the $\partial_{\overline{z}}$ operator, we get that $\partial_{\overline{z}} u = g\partial_{\overline{z}}\chi$. Hence $\partial_{\overline{z}} B_1 g = 0$ and $\partial_{\overline{z}} B_2 g = 0$. That is, $B_1 g$ and $B_2 g$ are holomorphic.

We finally obtain the estimates of $\|B_i\|$:

$$\begin{aligned} \|B_1 g\|_{L^p(\Omega_1)} &\leq \|g\|_{L^p(\Omega_1\cap\Omega_2)} + \left\|\frac{1}{z}\right\|_{L^1(B(0,2R))}\|g\partial_{\overline{z}}\chi\|_{L^p(\Omega_1\cap\Omega_2)} \\ &\leq \left(1 + \left\|\frac{1}{z}\right\|_{L^1(B(0,2R))} \times \frac{C}{\mathrm{d}(\Omega_1 \setminus \Omega_2,\ \Omega_2 \setminus \Omega_1)}\right)\|g\|_{L^p(\Omega_1\cap\Omega_2)}, \end{aligned}$$

and similarly for B_2. □

We can now prove theorem 2.2.

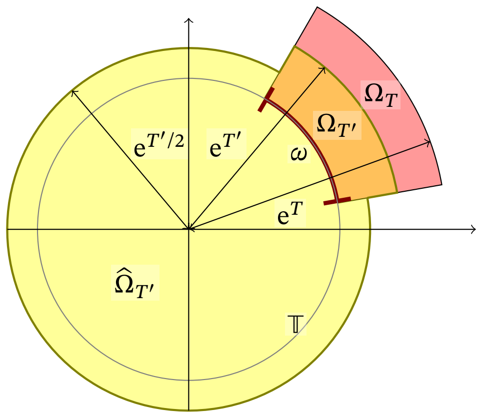

Figure 3: Illustration of $\widehat{\Omega}_{T'}$ and Ω_T when $T' < T$. The partial ring Ω_T is in red (including the reddish-yellow part). $\widehat{\Omega}_{T'}$ is in yellow (also including the reddish-yellow part). The intersection of Ω_T and $\widehat{\Omega}_{T'}$ is $\Omega_{T'}$ colored in reddish-yellow.

Proof of theorem 2.2. Suppose that the function f_0 is steerable to zero in time $T > 0$. According to theorem 1.2, this means that there exists $C_{f_0,T} > 0$ such that for every polynomial g,

$$|\langle f_0, g\rangle_{H^2}| \le C_{f_0,T}\|g\|_{A^2(\Omega_T)}. \tag{16}$$

Now, if $T' > T$, then $\Omega_T \subset \Omega_{T'}$, so that trivially $\|g\|_{A^2(\Omega_T)} \le \|g\|_{A^2(\Omega_{T'})}$ for $g \in A^2(\Omega_{T'})$, which in particular holds for polynomials. We thus get (16) for the norm in $A^2(\Omega_{T'})$ with the same constant $C_{f_0,T}$, showing that f_0 is null controllable in time $T' > T$ with control of norm at most $C_{f_0,T}$.

Consider the situation when $0 < T' < T$. We have to check the observability inequality (16) holds for $\Omega_{T'}$ and for every polynomial g knowing that it holds for functions in $A^2(\Omega_T)$. Set $\widehat{\Omega}_{T'} = e^{T'/2}\mathbb{D} \cup \Omega_{T'}$, which compactly contains the closure of $\mathbb{D}$, see fig. 3.

Then

$$\Omega_{T'} = \widehat{\Omega}_{T'} \cap \Omega_T.$$

By construction, $\widehat{\Omega}_{T'} \setminus \Omega_T$ and $\Omega_T \setminus \widehat{\Omega}_{T'}$ are at a strictly positive distance $e^{T'} - e^{T'/2} \ge T'/2$, and so, using Theorem 2.3, we have

$$A^2(\Omega_{T'}) = A^2(\widehat{\Omega}_{T'}) + A^2(\Omega_T).$$

Moreover, for every $g \in A^2(\Omega_{T'})$, we have $g = B_1 g + B_2 g$, where $B_1 g \in A^2(\widehat{\Omega}_{T'})$ and $B_2 g \in A^2(\Omega_T)$ and for some $C_T > 0$,

$$\|B_1 g\|_{A^2(\widehat{\Omega}_{T'})} \le \frac{C_T}{T'}\|g\|_{A^2(\Omega_{T'})},$$
$$\|B_2 g\|_{A^2(\Omega_T)} \le \frac{C_T}{T'}\|g\|_{A^2(\Omega_{T'})}.$$

Pick now a polynomial p, and let $p = B_1 p + B_2 p$ be a decomposition of p. Note that $B_1 p$ and $B_2 p$ need not to be polynomials. Define an anti-linear functional by

$$\ell_{f_0}(p) = \langle f_0, p\rangle_{H^2}, \quad p \text{ polynomial}.$$

We have to show that $|\ell_{f_0}(p)| \lesssim T'^{-3/2}\|p\|_{A^2(\Omega_{T'})}$ for every polynomial p.

Now, since f_0 is null controllable in time T, by the observability inequality (5) we have for every polynomial p

$$|\ell_{f_0}(p)| = |\langle f_0, p\rangle_{H^2}| \le C_{f_0,T}\|p\|_{A^2(\Omega_T)}.$$

Since polynomials are dense in $A^2(\Omega_T)$ [11, Section 18.1, Theorem 1.11][1] we can extend ℓ_{f_0} to a unique bounded functional ℓ_1 defined on the whole space $A^2(\Omega_T)$. We have $\|\ell_1\| \le C_{f_0,T}$.

On the other hand, since $\mathbb{D}^{\mathrm{cl}} \subset D(0, \mathrm{e}^{T'/2}) \subset \widehat{\Omega}_{T'}$, an elementary computation leads to

$$\begin{aligned}
\|g\|_{H^2}^2 &= \sum_{n\ge 0} |\widehat{g}(n)|^2 \\
&= \frac{2}{T'} \sum_{n\ge 0} |\widehat{g}(n)|^2 \frac{T'}{2} \\
&\le \frac{2}{T'} \sum_{n\ge 0} |\widehat{g}(n)|^2 \frac{\mathrm{e}^{T'(n+1)} - 1}{2n+2} \\
&= \frac{1}{T'} \|g\|^2_{A^2(\mathrm{e}^{T'/2}\mathbb{D}\setminus\mathbb{D}^{\mathrm{cl}})} \\
&\le \frac{1}{T'} \|g\|^2_{A^2(\widehat{\Omega}_{T'})}.
\end{aligned}$$

Hence, the Cauchy-Schwarz inequality yields

$$\begin{aligned}
|\ell_{f_0}(p)|^2 = |\langle f_0, p\rangle_{H^2}|^2 &= \left|\int_{\mathbb{T}} f_0(\mathrm{e}^{\mathrm{i}x})\overline{p(\mathrm{e}^{\mathrm{i}x})}\,\mathrm{d}x\right|^2 \le \|f_0\|^2_{H^2} \int_{\mathbb{T}} |p(\mathrm{e}^{\mathrm{i}x})|^2\,\mathrm{d}x \\
&\lesssim T'^{-1} \|p\|^2_{A^2(\widehat{\Omega}_{T'})}.
\end{aligned}$$

Analogously, since polynomials are dense in $A^2(\widehat{\Omega}_{T'})$ we can extend ℓ_{f_0} to a unique functional ℓ_2 defined on the whole space $A^2(\widehat{\Omega}_{T'})$, and $\|\ell_2\| \lesssim T'^{-1/2}\|f_0\|_{H^2}$.

Since $A^2(\Omega_T) \cap A^2(\widehat{\Omega}_{T'})$ $(= A^2(\Omega_T \cup \widehat{\Omega}_{T'}))$ is dense in $A^2(\Omega_{T'})$ (they all contain densely polynomials), we easily check that $\ell_1 = \ell_2$ on $A^2(\Omega_T) \cap A^2(\widehat{\Omega}_{T'})$. Hence the formula

$$\forall (g_1, g_2) \in A^2(\Omega_T) \times A^2(\widehat{\Omega}_{T'}),\ \ell(g_1 + g_2) := \ell_1(g_1) + \ell_2(g_2)$$

defines uniquely a functional on $A^2(\Omega_T) + A^2(\widehat{\Omega}_{T'})$. Clearly, by uniqueness, $\ell = \ell_1$ on $A^2(\Omega_T)$ and $\ell = \ell_2$ on $A^2(\widehat{\Omega}_{T'})$. Hence, for every polynomial p

$$|\ell_{f_0}(p)| = |\ell(p)| = |\ell(B_1 p + B_2 p)| \le |\ell(B_1 p)| + |\ell(B_2 p)| = |\ell_1(B_1 p)| + |\ell_2(B_2 p)|.$$

Now, since the ℓ_i are bounded on the underlying spaces we get

$$|\ell_{f_0}(p)| \lesssim \|B_1 p\|_{A^2(\widehat{\Omega}_{T'})} + T'^{-1/2}\|B_2 p\|_{A^2(\Omega_T)} \lesssim T'^{-3/2}\|p\|_{A^2(\Omega_{T'})},$$

which concludes the proof. □

2.3 Necessary conditions and sufficient conditions *via* complex analysis

We next prove the extension result. Since we have established the independence of null-controllability in time, we will henceforth no longer use the index T in Ω_T and simply write Ω to not overcharge notation. Recall that given $\Omega = \{r\mathrm{e}^{\mathrm{i}t},\ \mathrm{e}^{\mathrm{i}t} \in \omega,\ 1 < r < \mathrm{e}^T\}$ (domain outside $\mathbb{D}$) we denote $\Omega^* = \{r\mathrm{e}^{\mathrm{i}t},\ \mathrm{e}^{\mathrm{i}t} \in \omega,\ \mathrm{e}^{-T} < r < 1\}$ (domain inside $\mathbb{D}$).

[1] Note that all sets on which we consider Bergman spaces in this paper are so-called Carathéodory domains, see [11, Section 18.1, Definition 1.6 and Proposition 1.9] which gives density of polynomials in A^2.

Proof of theorem 1.4. Step 1: Holomorphic extension. — For $u \in \mathbb{D}$, let $k_u(z) = \frac{1}{2\pi(1-\overline{u}z)}$ be the reproducing kernel of H^2, which means that for every $f \in H^2(\mathbb{D})$, $\langle f, k_u \rangle_{H^2} = f(u)$. The function k_u extends to a holomorphic function in Ω for every $u \in \mathbb{C} \setminus \Omega^*$, and it is clear that for every $u \in \mathbb{C} \setminus (\Omega^*)^{\mathrm{cl}}$, we have $k_u \in A^2(\Omega)$. Recall the anti-linear functional introduced previously

$$\ell_{f_0}(p) = \langle f_0, p \rangle_{H^2}, \quad p \text{ polynomial.}$$

Polynomials being dense in $A^2(\Omega)$, ℓ_{f_0} is well defined and continuous on $A^2(\Omega)$. Since $k_u \in A^2(\Omega)$ for $u \in \mathbb{C} \setminus (\Omega^*)^{\mathrm{cl}}$, we can define

$$\phi(u) = \ell_{f_0}(k_u).$$

Now, the function $u \in \mathbb{C} \setminus (\Omega^*)^{\mathrm{cl}} \mapsto k_u \in A^2(\Omega)$ is obviously anti-analytic, and so the function ϕ is analytic on $\mathbb{C} \setminus (\Omega^*)^{\mathrm{cl}}$. Since k_u is the reproducing kernel of H^2, we get for $|u| < 1$ and $u \notin (\Omega^*)^{\mathrm{cl}}$, $\phi(u) = \ell_{f_0}(u) = \langle f_0, k_u \rangle_{H^2} = f_0(u)$. Hence, since f_0 is holomorphic in $\mathbb{D}$, f_0 extends holomorphically to $\mathbb{C} \setminus \omega^{cl}$.

Another consequence of the continuity of ℓ_{f_0} on $A^2(\Omega)$ is the existence of $\psi \in A^2(\Omega)$ such that $\ell_{f_0}(p) = \langle \psi, p \rangle_{A^2(\Omega)}$ and hence

$$f_0(u) = \ell_{f_0}(k_u) = \langle \psi, k_u \rangle_{A^2}.$$

Step 2: Estimate for $|u| \to \infty$. — Since $\|k_u\|_{A^2(\Omega)} \leq C/|u|$ as $|u| \to, \infty$, we get

$$|f_0(u)| \leq C\|\ell_{f_0}\|\frac{1}{|u|}.$$

This proves that f_0 extends to ∞ with $f_0(\infty) = 0$.

Step 3: Regularity. — We denote the reproducing kernel of $A^2(\Omega)$ by k_u^Ω [16, §1.1]. Recall the following usual transformation formula of the reproducing kernel (see for instance [16, §1.3, Theorem 3])

$$k_\lambda^{\Omega_1}(z) = k_{\varphi(\lambda)}^{\Omega_2}(\varphi(z))\varphi'(z)\overline{\varphi'(\lambda)}, \quad z, \lambda \in \Omega_1, \tag{17}$$

where $\varphi\colon \Omega_1 \to \Omega_2$ is the conformal mapping from Ω_1 to Ω_2. Denoting as before by $\mathbb{D}_\mathrm{e} = \{z \in \widehat{\mathbb{C}},\ |z| > 1\}$ the outer disk, we get with $\varphi\colon \mathbb{D}_\mathrm{e} \to \mathbb{D}$, $\varphi(z) = 1/z$, and $k_u^{\mathbb{D}}(z) = 1/\big(\pi(1-\overline{u}z)^2\big)$:

$$k_u^{\mathbb{D}_\mathrm{e}}(z) = \frac{1}{\pi(1-\overline{u}z)^2}, \quad |u| > 1, |z| > 1.^{2}$$

We denote $P_{A^2(\mathbb{D}_\mathrm{e})}\colon h \in L^2(\mathbb{D}_\mathrm{e}) \to L^2(\mathbb{D}_\mathrm{e})$ the orthogonal projection onto $A^2(\mathbb{D}_\mathrm{e})$, which is given by $P_{A^2(\mathbb{D}_\mathrm{e})}f(u) = \langle f, k_u^{\mathbb{D}_\mathrm{e}} \rangle_{L^2(\mathbb{D}_\mathrm{e})}$ (see [50, Lemma 2.8] for the Bergman space on the disk, but the proof is the same for any domain).

Remark that $\partial_{\overline{u}} k_u(z) = 2^{-1}z k_u^{\mathbb{D}_\mathrm{e}}(z)$. Hence, if we differentiate the formula $f_0(u) = \langle \psi, k_u \rangle_{A^2(\Omega)}$ in u, $|u| > 1$, we get by Lebesgue's dominated convergence theorem

$$\begin{aligned} f_0'(u) &= \langle \psi, \partial_{\overline{u}} k_u \rangle_{A^2(\Omega)} \\ &= \left\langle \psi, \frac{z}{2} k_u^{\mathbb{D}_\mathrm{e}} \right\rangle_{A^2(\Omega)} \\ &= \left\langle \frac{\overline{z}}{2}\psi \mathbb{1}_\Omega, k_u^{\mathbb{D}_\mathrm{e}} \right\rangle_{L^2(\mathbb{D}_\mathrm{e})} \\ &= P_{A^2(\mathbb{D}_\mathrm{e})}\left(\frac{\overline{z}}{2}\psi \mathbb{1}_\Omega\right)(u). \end{aligned}$$

The function $z \mapsto \overline{z}\psi(z)\mathbb{1}_\Omega(z)$ is in $L^2(\mathbb{D}_\mathrm{e})$ because $\psi \in A^2(\Omega)$ and Ω is bounded. We deduce that $f_0' \in A^2(\mathbb{D}_\mathrm{e})$. □

[2] Observe that this is the same formula as for $k_u^{\mathbb{D}}$, $|u| < 1$.

Proof of theorem 1.6. Assume that $f_0 \in \mathcal{NC}_{H^2}(\omega)$. By theorem 1.4, f_0 extends holomorphically to $\widehat{\mathbb{C}} \setminus \omega^{\mathrm{cl}}$ (theorem 1.4). Moreover, denoting again this extension by f_0, we have $f_0(\infty) = 0$.

Now, assuming that f_0 also extends holomorphically to $D(0, 1+\epsilon)$, then f_0 is actually holomorphic on $\widehat{\mathbb{C}}$. Since $f_0(\infty) = 0$, Liouville's theorem implies that $f_0 = 0$. □

Proof of theorem 1.5. Let $f_0 \in \mathcal{NC}_{H^2}(\omega)$. According to theorem 1.4, f_0 extends as a holomorphic functions of $\widehat{\mathbb{C}} \setminus \omega^{\mathrm{cl}}$ such that $f_0(\infty) = 0$ and $f_0 \in \mathcal{D}(\mathbb{D}_{\mathrm{e}})$

For a finite Borel measure μ on $\mathbb{T}$, we denote by $C\mu$ the *Cauchy transform* of μ defined for $z \in \mathbb{C} \setminus \mathbb{T}$ by

$$C\mu(z) = \int_{\mathbb{T}} \frac{\mathrm{d}\mu(\zeta)}{1 - \bar{\zeta} z}.$$

Step 1: There exists $g \in L^2(\mathbb{T})$ *such that* $f_0 = Cg$. — (When writing Cg, we mean $C\mu$, where $\mathrm{d}\mu(u) = g(u)\,\mathrm{d}u/2\pi$.) Tumarkin's theorem [10, Theorem 5.3.1] *almost* gives us this result. To actually get what we want, we find it is easier to adapt the proof of Tumarkin's theorem.

Since f_0 extends analytically to $\widehat{\mathbb{C}} \setminus \omega^{\mathrm{cl}}$ we can write for $|u| < 1$, $f_0(u) = \sum_{n \geq 0} a_n u^n$, and for $|u| > 1$, $f_0(u) = \sum_{n<0} a_n u^n$ (there is no a_0 in this latter sum because $f_0(\infty) = 0$). Since $f_0 \mathbb{1}_{\mathbb{D}} = f_0 \in H^2$, we have $a_n = \widehat{f_0}(n)$, $n \geq 0$, so that $(a_n)_{n \geq 0} \in \ell^2$. And since $f_0 \mathbb{1}_{\mathbb{D}_{\mathrm{e}}} \in \mathcal{D}$, we have $(\sqrt{n} a_n)_{n<0} \in \ell^2$. We can thus define g a.e. on $\mathbb{T}$ as the $L^2(\mathbb{T})$-function

$$g(\zeta) = \sum_{n \geq 0} a_n \zeta^n - \sum_{n<0} a_n \zeta^n, \quad \text{for almost every } |\zeta| = 1.$$

As in (6) we have

$$\int_{\mathbb{T}} \frac{g(w)}{2\pi(1 - \overline{w} z)} |\mathrm{d}w| = \begin{cases} P_+ g(z) = \sum_{n \geq 0} \widehat{g}(n) z^n = \sum_{n \geq 0} a_n z^n, & |z| < 1, \\ -P_- g(z) = -\sum_{n>0} \dfrac{\widehat{g}(-n)}{z^n} = \sum_{n>0} \dfrac{a_{-n}}{z^n}, & |z| > 1, \end{cases}$$

which shows that $f_0 = Cg$ on $\widehat{\mathbb{C}} \setminus \omega^{\mathrm{cl}}$, and in particular $f_0 = P_+ g$ in $\mathbb{D}$.

Step 2: $\mathrm{supp}(g) \subset \omega^{\mathrm{cl}}$. — According to Fatou's jump theorem [10, Corollary 2.4.2], for almost every $u \in \mathbb{T}$,

$$g(u) = \lim_{r \to 1^-} (Cg(ru) - Cg(u/r)) = \lim_{r \to 1^-} (f_0(ru) - f_0(u/r)).$$

But f_0 is holomorphic outside ω^{cl}. Hence, if $u \in \mathbb{T} \setminus \omega^{\mathrm{cl}}$, the above limit is 0, hence $\mathrm{supp}\, g \subset \omega^{\mathrm{cl}}$. □

We next prove theorem 1.7. This will be a consequence of a necessary and sufficient condition related to pseudo-Carleson measures (see theorem 2.5 below). We will first need some notation. We refer to fig. 4 for the following construction. We denote the endpoints of ω by ζ_1 and ζ_2. Let $0 < \eta < \eta_0 < T$ and pick ω_0 an arc compactly contained in ω with endpoints ζ_1' and ζ_2' having same center as ω. In order to avoid pathological situations, we will assume that ζ_k' is sufficiently close to ζ_k (for instance $\eta_0 \pi/8 \leq |\zeta_k' - \zeta_k| \leq \eta_0 \pi/4$). Introduce the segments $\Gamma_k = [\zeta_k, e^{\eta_0} \zeta_k']$, $k = 1, 2$, and define $\Gamma_{k,\eta}$ to be the part of Γ_k in $|z| \geq (1+\eta)$. We also let ω_η to be the arc in $(1+\eta)\mathbb{T}$ comprised between the segments Γ_1 and Γ_2. Define $\widetilde{\Omega}_\eta$ as the set bounded by ω_η, $\Gamma_{1,\eta}$, $\Gamma_0 \coloneqq \omega_{\eta_0}$ and $\Gamma_{2,\eta}$. In particular

$$\partial \widetilde{\Omega}_\eta = \omega_\eta \cup \Gamma_{1,\eta} \cup \Gamma_0 \cup \Gamma_{2,\eta}.$$

In view of a future application of Stokes' theorem we note that $\widetilde{\Omega}_\eta$ has regular boundary (in the sense of [4, Définition 2.3.1]). Observe also that Γ_0 is at a uniform distance from $\partial \Omega_T^*$.

We also let $\widetilde{\Omega} = \bigcup_{0<\eta<\eta_0} \widetilde{\Omega}_\eta$. Hence

$$\partial \widetilde{\Omega} = \omega \cup \Gamma_1 \cup \Gamma_0 \cup \Gamma_2.$$

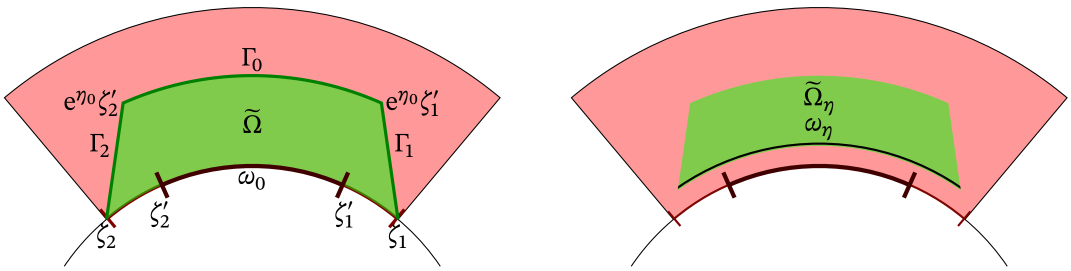

Figure 4: Left figure: illustration of $\widetilde{\Omega}$ (in green), with Ω in red, the paths Γ_i ($i = 0, 1, 2$) and the points ζ_i' and $e^{\eta_0}\zeta_i'$ ($i = 1, 2$). Right figure: illustration of $\widetilde{\Omega}_\eta$ (in green) and the path ω_η.

Theorem 2.5. *The function $f_0 \in H^2$ is in $\mathcal{NC}_{H^2}(\omega)$ if and only if there exists $g \in L^2(\mathbb{T})$, such that*

1. $f_0 = P_+ g$,
2. $\operatorname{supp} g \subset \omega^{\mathrm{cl}}$,
3. $\sum_{n<0} |n||\widehat{g}(n)|^2 < +\infty$,
4. *Denoting by G the harmonic extension of $\zeta \to \zeta g(\zeta)$ to $\mathbb{D}_e$ the measure $\overline{G}\, \mathrm{d}\zeta$ on $\Gamma_1 \cup \Gamma_2$ is pseudo-Carleson for $A^2(\Omega)$, i.e.*
$$\left|\int_{\Gamma_1\cup\Gamma_2} \overline{G}(\zeta)p(\zeta)\,\mathrm{d}\zeta\right| \lesssim \|p\|_{A^2(\Omega)}. \tag{18}$$

As we will see in the proof, $G(z) = g_1(1/\overline{z}) + g_2(z)$, $|z| > 1$, with $g_2 \in \mathcal{D}(\mathbb{D}_e)$ and $g_1 \in H^2$. In particular, G can be written as $G(z) = \sum_{n\geq 0} G_n \overline{z}^{-n} + \sum_{n<0} G_n z^{-|n|}$ for some $(G_n)_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$. Let us show that this implies that the integral on the left hand side of (18) is well defined. By rotation, we can assume without loss of generality that Γ_i is a path of the form $s \in (0, \epsilon) \mapsto 1 + se^{i\theta}$, for some $\theta \in (-\pi/2, \pi/2)$. Thanks to Fubini's theorem,

$$\begin{aligned}
\Big|\int_{\Gamma_i} \overline{G(\zeta)}p(\zeta)\,\mathrm{d}\zeta\Big| &= \Big|\int_{\Gamma_i}\Big(\sum_{n\geq 0}\overline{G_n}\zeta^{-n} + \sum_{n<0}\overline{G_n}\overline{\zeta}^{-|n|}\Big)p(\zeta)\,\mathrm{d}\zeta\Big| \\
&\lesssim \sum_{n\in\mathbb{Z}}\int_0^\epsilon |G_n||1+se^{i\theta}|^{-|n|}\|p\|_{\infty,\mathbb{D}}\,\mathrm{d}s \\
&\lesssim \sum_{n\in\mathbb{Z}}\int_0^\epsilon |G_n|(1+c_\theta s)^{-|n|}\|p\|_{\infty,\mathbb{D}}\,\mathrm{d}s \\
&\lesssim \|p\|_{\infty,\mathbb{D}}\sum_{n\in\mathbb{Z}}\frac{|G_n|}{|n|+1} \\
&\lesssim \|p\|_{\infty,\mathbb{D}}\Big(\sum_{n\in\mathbb{Z}}\frac{1}{(|n|+1)^2}\Big)^{1/2}\Big(\sum_{n\in\mathbb{Z}}|G_n|^2\Big)^{1/2}.
\end{aligned}$$

In particular, we have

$$\int_{\Gamma_1\cup\Gamma_2}\overline{G}(\zeta)p(\zeta)\,\mathrm{d}\zeta = \lim_{\eta\to 0}\int_{\Gamma_{1,\eta}\cup\Gamma_{2,\eta}}\overline{G}(\zeta)p(\zeta)\,\mathrm{d}\zeta.$$

Proof. *Step 1: Poisson kernel on* $\mathbb{D}_e$. — Let $P^e[h]$ be the harmonic extension (Poisson integral) of a function $h \in L^2(\mathbb{T})$ to the outer disk $\mathbb{D}_e$ (see for instance [5, Theorem 4.11]):

$$P^e[h](z) = \frac{1}{2\pi}\int_{\mathbb{T}} P^e(z,\zeta)h(\zeta)|\mathrm{d}\zeta|, \quad |z| > 1,$$

where

$$P^e(z,\zeta) = \frac{|z|^2-1}{|z-\zeta|^2}, \quad |\zeta| = 1,\ |z| > 1.$$

Keeping in mind that for $|\zeta| = 1$, $h(\zeta) = \sum_n \widehat{h}(n)\zeta^n$, we claim that,

$$P^e[h](z) = \sum_{n\geq 0}\widehat{h}(n)\frac{1}{\overline{z}^n} + \sum_{n>0}\widehat{h}(-n)\frac{1}{z^n}.$$

Indeed, an elementary computation shows that $P^e(z,\mathrm{e}^{\mathrm{i}t}) = 1 + \sum_{n>0} z^{-n}\mathrm{e}^{\mathrm{i}nt} + \overline{z}^{-n}\mathrm{e}^{-\mathrm{i}nt}$, and a computation using the orthogonality of the complex exponentials gives the claimed formula.

Step 2: Use of Stokes' formula. — Consider now $g(\zeta) = \sum_{n\in\mathbb{Z}}\widehat{g}(n)\zeta^n \in L^2(\mathbb{T})$. We denote for $|z| > 1$, $g_1(z) = \sum_{n\geq 0}\widehat{g}(n)\overline{z}^{-n}$ that is anti-holomorphic and $g_2(z) = \sum_{n>0}\widehat{g}(-n)z^{-n}$ that is holomorphic. In our argument below we need to consider zg rather than g (this will be needed in the change of variable $\mathrm{d}z = \mathrm{i}z\,\mathrm{d}t$ for $z = c\mathrm{e}^{\mathrm{i}t}$, where c is a suitable constant), but clearly this does not affect our decomposition and we can actually write for $\zeta \in \mathbb{T}$, $\zeta g(\zeta) = g_1(\zeta) + g_2(\zeta)$ with the same form as above (it is just a shift of indices).

Now, set $G = P^e[zg]$ and let us apply Stokes' formula to the form $w = \overline{G}p\,\mathrm{d}z$ on the domain $\widetilde{\Omega}_\eta$. Since $\widetilde{\Omega}_\eta$ is compactly contained in $\mathbb{D}_e$, all the functions g, g_1, g_2 are C^∞ on this domain and we can apply Stokes' formula there.

$$\int_{\partial\widetilde{\Omega}_\eta} w(z) = \int_{\widetilde{\Omega}_\eta}\mathrm{d}w(z) = \int_{\widetilde{\Omega}_\eta}\partial_{\overline{z}}(\overline{P^e[zg]}p)\,\mathrm{d}\overline{z}\wedge\mathrm{d}z = 2\mathrm{i}\int_{\widetilde{\Omega}_\eta}\overline{g_2'}p\,\mathrm{d}A(z). \tag{19}$$

Observe that the anti-analytic function $g_1(z)$ disappears when differentiating in z.

Decomposing $\partial\widetilde{\Omega}_\eta$ this gives

$$\int_{\partial\widetilde{\Omega}_\eta} w(z) = \int_{\omega_\eta}\overline{G}(z)p(z)\,\mathrm{d}z + \int_{\Gamma_{1,\eta}\cup\Gamma_{2,\eta}}\overline{G}(z)p(z)\mathrm{d}z + \int_{\Gamma_0}\overline{G}(z)p(z)\mathrm{d}z = \int_{\widetilde{\Omega}_\eta}\overline{g_2'}(z)p(z)\,\mathrm{d}A(z).$$

Step 3: Convergence of the integral for $\eta \to 0$. — Let us show that if $g_2' \in \mathcal{D}(\mathbb{D}_e)$, then

$$\int_{\partial\widetilde{\Omega}}\overline{G}(z)p(z)\,\mathrm{d}z = 2\mathrm{i}\int_{\widetilde{\Omega}}\overline{g_2'}(z)p(z)\,\mathrm{d}A(z). \tag{20}$$

Recall that $g_2' \in L^2(\mathbb{D}_e)$ if and only if $\sum_{n<0}|n||\widehat{g}(n)|^2 < +\infty$.

Considering the right-hand side, we assumed that $g_2' \in L^2(\mathbb{D}_e)$, hence, according to Lebesgue's dominated convergence theorem,

$$\int_{\widetilde{\Omega}_\eta}\overline{g_2'}(z)p(z)\,\mathrm{d}A(z) \xrightarrow[\eta\to 0]{} \int_{\widetilde{\Omega}}\overline{g_2'}(z)p(z)\,\mathrm{d}A(z).$$

To treat the left-hand side, we start with Fubini's theorem:

$$\int_{\omega_\eta}\overline{G(z)}p(z)\,\mathrm{d}z = \int_{\omega_\eta}\int_{\mathbb{T}}\frac{|z|^2-1}{|z-\zeta|^2}\overline{\zeta g(\zeta)}|\mathrm{d}\zeta|p(z)\,\mathrm{d}z = \int_{\mathbb{T}}\overline{\zeta g(\zeta)}\int_{\omega_\eta}\frac{|z|^2-1}{|z-\zeta|^2}p(z)\,\mathrm{d}z|\mathrm{d}\zeta|.$$

Let now $\tilde{\omega}_\eta$ be the projection of ω_η onto $\mathbb{T}$ (this is actually the intersection of the convex hull of 0 and ω_η with $\mathbb{T}$). Then, with the change of variable $z = (1+\eta)\mathrm{e}^{\mathrm{i}t}$, so that $\mathrm{d}z = (1+\eta)\mathrm{i}\mathrm{e}^{\mathrm{i}t}\,\mathrm{d}t = \mathrm{i}z\,\mathrm{d}t$, and $\zeta = \mathrm{e}^{\mathrm{i}s}$, we get

$$\varphi_\eta(\zeta) := \int_{\omega_\eta} \frac{|z|^2-1}{|z-\zeta|^2} p(z)\,\mathrm{d}z = \int_{\tilde{\omega}_\eta} \frac{(1+\eta)^2-1}{|(1+\eta)\mathrm{e}^{\mathrm{i}(t-s)}-1|^2}(1+\eta)\mathrm{i}\mathrm{e}^{\mathrm{i}t} p((1+\eta)\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}t \to \begin{cases} \mathrm{i}\zeta p(\zeta) & \zeta \in \omega, \\ 0 & \zeta \notin \omega^{\mathrm{cl}}. \end{cases}$$

Indeed, it is clear that if $\zeta = \mathrm{e}^{\mathrm{i}s} \notin \omega^{\mathrm{cl}}$, then the Poisson kernel goes uniformly to 0 on $\tilde{\omega}_\eta$. When $\zeta = \mathrm{e}^{\mathrm{i}s} \in \omega$, then for a sufficiently small $\eta > 0$, we have $\zeta \in \tilde{\omega}_\eta$. Since the Poisson kernel is an approximate identity, it will concentrate in a neighborhood of ζ when $\eta \to 0$, so that in this case, the integral will converge to $\mathrm{i}\zeta p(\zeta)$ (note that this function is a polynomial). Morever,

$$|\varphi_\eta(\zeta)| = \Big|\int_{\omega_\eta} \frac{|z|^2-1}{|z-\zeta|^2} p(z)\,\mathrm{d}z\Big| \lesssim \sup_{z\in\omega_\eta} |p(z)| \int_{\omega_\eta} \frac{|z|^2-1}{|z-\zeta|^2}|\mathrm{d}z| \le \|p\|_{\infty,\mathbb{T}} \int_{\mathbb{T}} \frac{|z|^2-1}{|z-\zeta|^2}|\mathrm{d}z| = \|p\|_{\infty,\mathbb{T}},$$

so that Lebesgue's dominated convergence theorem yields

$$\lim_{\eta\to 0} \int_{\omega_\eta} \overline{G(z)} p(z)\,\mathrm{d}z = \int_\omega \overline{g}(\zeta) p(\zeta)|\mathrm{d}\zeta|. \tag{21}$$

Together with Stokes' formula (19), we get the desired formula eq. (20).

Step 4: Necessary condition. — We already know from theorem 1.5 that the conditions 1.–3. are necessary. Let g be given by theorem 1.5 and $G = P^e[zg]$. The preceding equality yields

$$\Big|\int_{\Gamma_1\cup\Gamma_2} \overline{G}(z)p(z)\,\mathrm{d}z\Big| \le \Big|\int_\omega \overline{G}(z)p(z)\,\mathrm{d}z\Big| + \Big|\int_{\Gamma_0} \overline{G}(z)p(z)\,\mathrm{d}z\Big| + \Big|\int_{\tilde{\Omega}} \overline{g_2'}(z)p(z)\,\mathrm{d}A(z)\Big|. \tag{22}$$

Since $g_2 \in \mathcal{D}(\mathbb{D}_e)$, the Cauchy-Schwarz inequality yields

$$\Big|\int_{\tilde{\Omega}} \overline{g_2'}(z)p(z)\,\mathrm{d}A(z)\Big| \le \|g_2\|_{\mathcal{D}(\mathbb{D}_e)}\|p\|_{A^2(\Omega)}.$$

Recall that the norm of the point evaluation in $A^2(\Omega)$ is essentially controlled by the inverse of the distance to the boundary [11, Chapter 18, Proposition 1.3]. Since Γ_0 is compactly contained in Ω hence g is uniformly bounded on Γ_0, and the integral on Γ_0 is controlled by $\|p\|_{A^2(\Omega)}$.

By the assumption that f_0 is null-controllable, the modulus of the integral on ω in eq. (22) is controlled by $\|p\|_{A^2(\Omega)}$. Hence, eq. (22) shows that for every polynomial p,

$$\Big|\int_{\Gamma_1\cup\Gamma_2} \overline{G}(z)p(z)\,\mathrm{d}z\Big| \lesssim \|p\|_{A^2(\Omega)}.$$

Step 5: Sufficient condition. — Suppose f_0 satisfies the 4 conditions of the theorem. Stokes' formula (eq. (20)) yields

$$\Big|\int_{\omega_\eta} \overline{G}(z)p(z)\,\mathrm{d}z\Big| \le \Big|\int_{\Gamma_{1,\eta}\cup\Gamma_{2,\eta}} \overline{G}(z)p(z)\,\mathrm{d}z\Big| + \Big|\int_{\Gamma_0} \overline{G}(z)p(z)\,\mathrm{d}z\Big| + \Big|\int_{\tilde{\Omega}_\eta} \overline{g_2'}p\,\mathrm{d}A(z)\Big|.$$

The last two terms are again controlled by $\|p\|_{A^2(\Omega)}$ using the same arguments as in the necessary condition (note that $g \in H^2 + \mathcal{D}(\mathbb{D}_e)$ is our standing hypothesis). Moreover, we have by assumption $|\int_{\Gamma_1\cup\Gamma_2} \overline{G}(z)p(z)\,\mathrm{d}z| \lesssim \|p\|_{A^2(\Omega)}$, from where we deduce that

$$\Big|\int_\omega \overline{g}(\mathrm{e}^{\mathrm{i}t})p(\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}t\Big| \lesssim \|p\|_{A^2(\Omega)}.$$

showing that $P_+g \in \mathcal{NC}_{H^2}(\omega)$. □

We are now in a position to prove Theorem 1.7.

Proof of theorem 1.7. For this, it is enough to estimate the integral condition of theorem 2.5 on $\Gamma_1 \cup \Gamma_2$ under the assumption

$$\int_{\mathbb{T}} \frac{|g(e^{it})|^2}{|(e^{it}-\zeta_1)(e^{it}-\zeta_2)|}\,dt < \infty. \tag{23}$$

It is clearly enough to check that

$$\int_{\Gamma_1} |\overline{G}(z)p(z)||dz| \lesssim \|p\|_{A^2(\Omega)}.$$

Let us parameterize Γ_1. In order to simplify the argument we assume that ω starts at $\zeta = 1$ and that the segment Γ_1 takes the direction $(1+i)$. Then a parametrization is given by $\gamma(s) = 1 + s(1+i)$, $s \in [0, \delta_0]$ (we assume that ζ_1 is the lower left corner of Ω, and so Γ_1 moves north-east). In particular $|dz| = \sqrt{2}\,ds$. By the Cauchy-Schwarz inequality, we have

$$\int_{\Gamma_1} |\overline{G}(z)p(z)||dz| \lesssim \left(\int_0^{\delta_0} \frac{|G(\gamma(s))|^2}{s}\,ds\right)^{1/2} \left(\int_0^{\delta_0} s|p(\gamma(s))|^2\,ds\right)^{1/2}.$$

We claim that the second term translates to a Carleson measure result. Indeed, setting $d\mu(z) = s\,d_{\Gamma_1}(\gamma(s))$, where d_{Γ_1} is arclength measure on Γ_1, it is enough to check that $d\mu$ is a Carleson measure for $A^2(\Omega)$. For this, using a result by Gonzalez [22], we have to check that $\mu(B(z,r)) \lesssim r^2$ for Whitney balls in Ω_T, which are balls having a radius which is less than one half the distance of z to the boundary of Ω_T but still is comparable to that distance. Clearly, in the worst case such a ball contains Γ_1 as a diameter, in which case $\int_{B(z,r)\cap\Gamma_1} s\,ds \asymp r^2$.

In order to finish the proof we thus have to estimate the first term. By Jensen's inequality, we have

$$|G(\gamma(s))|^2 = \left(\int_{\mathbb{T}} \frac{|\gamma(s)|^2-1}{|\gamma(s)-e^{it}|^2} g(e^{it})\,dt\right)^2 \lesssim \int_{\mathbb{T}} \frac{|\gamma(s)|^2-1}{|\gamma(s)-e^{it}|^2} |g(e^{it})|^2\,dt.$$

Now, letting L be the length of ω (which we had chosen to start at $\zeta_1 = 1$), and since for $z = \gamma(s) \in \Gamma_1$ we have $|z|^2 - 1 \asymp s$ and $|e^{it} - z|^2 \asymp (s-t)^2 + s^2$, an application of Fubini's theorem gives

$$\int_0^{\delta_0} \frac{|G(\gamma(s))|^2}{s}\,ds \lesssim \int_0^L |g(e^{it})|^2 \int_0^{\delta_0} \frac{1}{(s-t)^2+s^2}\,ds\,dt. \tag{24}$$

Observing that $\int_0^{\delta_0} \frac{1}{(s-t)^2+s^2}\,ds = \frac{1}{2}\int_0^{\delta_0} \frac{1}{(s-t/2)^2+t^2/4}\,ds$, by a change of variables and standard estimates we control this integral by C/t (starting from the last integral, one can reach this estimate also by reinterpreting this integral, after multiplication by $t/2$, as a Poisson integral). By (23) we obtain the required control on the piece Γ_1, and similarly on Γ_2. □

To prove that $\mathcal{NC}_{H^2}(\omega)$ is dense, we will use the fact that P_+ is self-adjoint on $W^{k,2}(\mathbb{T})$. Recall also that $H^2 \cap W^{k,2}(\mathbb{T})$ is endowed with the $W^{k,2}$ scalar product given by

$$\langle f, g\rangle_{W^{k,2}(\mathbb{T})} := \sum_{n\in\mathbb{Z}} (1+n^2)^k \widehat{f}(n)\overline{\widehat{g}(n)}.$$

We will also use the following

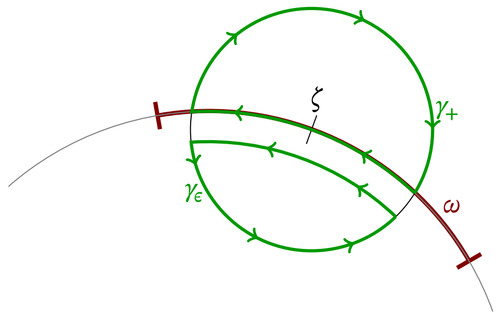

Figure 5: Illustration of the paths γ_ϵ and γ_+. They are used in the proof of theorem 2.6 to prove with Morera's theorem that the function $H(z) = f(z)$ if $|z| < 1$ and $H(z) = h(z)$ if $|z| \geq 1$ and $z \notin (-\infty, 0]$ is holomorphic.

Lemma 2.6. *Let $k \in \mathbb{N}$. Let $f \in H^2 \cap W^{k,2}(\mathbb{T})$. Assume that*

$$\forall g \in C_c^\infty(\omega),\ \langle f, g \rangle_{W^{k,2}(\mathbb{T})} = 0.$$

Then $f = 0$.

Proof. In this proof, it is useful to distinguish the maps $z \in \mathbb{D} \mapsto f(z)$ and $\widetilde{f} \colon x \in \mathbb{R} \mapsto f(\mathrm{e}^{\mathrm{i}x})$. In the same vein, we define $\widetilde{\omega} \subset [-\pi, \pi]$ by $\omega = \{\mathrm{e}^{\mathrm{i}x},\ x \in \widetilde{\omega}\}$. Up to a rotation, we may assume that $-1 \notin \omega^{\mathrm{cl}}$, in which case $\widetilde{\omega}$ is an interval of $[-\pi, \pi]$.

Step 1: $\widetilde{f}(x)$ is a linear combination of $x^\ell \mathrm{e}^{\pm x}$ $(0 \leq \ell < k)$ on $\widetilde{\omega}$. — Indeed, if $g \in C_c^\infty(\omega)$, the $W^{k,2}$ scalar product can be written as

$$\langle f, g \rangle_{W^{k,2}(\mathbb{T})} = \langle (I - \partial_x^2)^k \widetilde{f}, \widetilde{g} \rangle_{W^{-k,2}(-\pi,\pi), W_0^{k,2}(-\pi,\pi)}.$$

If this is 0 for every $g \in C_c^\infty(\omega)$ (i.e., for every $\widetilde{g} \in C_c^\infty(\widetilde{\omega})$), this means that $(I - \partial_x^2)^k \widetilde{f} = 0$ on $\widetilde{\omega}$ (by definition in the distribution sense). The functions $x \mapsto x^\ell \mathrm{e}^{\pm x}$ for $0 \leq \ell < k$ are a basis of solutions of this ODE, hence there exists $(a_\ell^\pm)_\ell$ such that $\forall x \in \widetilde{\omega}$,

$$\widetilde{f}(x) = \sum_{0 \leq \ell < k} \mathrm{i}^\ell x^\ell (a_\ell^+ \mathrm{e}^{-x} + a_\ell^- \mathrm{e}^{x}). \tag{25}$$

The i^ℓ is only here to simplify the notations in the following.

Step 2: Extension of the formula (25) to a larger domain. — We can define the holomorphic function h on $\mathbb{C} \setminus (-\infty, 0]$ by

$$h(z) := \sum_{0 \leq \ell < k} (\ln(z))^\ell (a_\ell^+ z^{\mathrm{i}} + a_\ell^- z^{-\mathrm{i}}). \tag{26}$$

Since $\widetilde{f}(x) = f(\mathrm{e}^{\mathrm{i}x})$, setting $z = \mathrm{e}^{\mathrm{i}x}$, we can identify $\mathrm{i}x = \ln(z)$ and $\mathrm{e}^{-x} = z^{\mathrm{i}}$. Hence, by direct inspection, we see that the non-tangential limits of f on ω coincide with the values of h on ω. This implies that f extends h through ω inside $\mathbb{D}$. To see this, one can for instance use Morera's theorem. Define $H(z) = f(z)$ if $|z| < 1$ and $H(z) = h(z)$ for $|z| \geq 1$, $z \notin (-\infty, 0]$. For $\zeta \in \omega$, let D_ζ be a circle centered at ζ such that $D_\zeta \cap \mathbb{T} \subset \omega$, and define $\gamma = \partial D_\zeta$. For every $\epsilon > 0$, we let $\gamma_\epsilon = (\gamma \cap (1-\epsilon)\mathbb{D}^{\mathrm{cl}}) \cup ((1-\epsilon)\mathbb{T} \cap D_\zeta)$ (see fig. 5). Define also $\gamma_+ = \partial(D_\zeta \cap \mathbb{D}_{\mathrm{e}}{}^{\mathrm{cl}})$. Since $f \in H(\mathbb{D})$

and $h \in H(\mathbb{D}_e)$, $\int_{\gamma_\epsilon} H(z)\,\mathrm{d}z = \int_{\gamma_\epsilon} f(z)\,\mathrm{d}z = 0$, and $\int_{\gamma_+} H(z)\,\mathrm{d}z = \int_{\gamma_+} g(z)\,\mathrm{d}z = 0$. Moreover, $f \in H^2$ implies that

$$\int_{(1-\epsilon)\mathbb{T}\cap D_\zeta} H(z)\,\mathrm{d}z = \int_{(1-\epsilon)\mathbb{T}\cap D_\zeta} f(z)\,\mathrm{d}z \xrightarrow[\epsilon\to 0]{} \int_{D_\zeta\cap\mathbb{T}} f(z)\,\mathrm{d}z = \int_{D_\zeta\cap\mathbb{T}} H(z)\,\mathrm{d}z.$$

Hence

$$\int_\gamma H(z)\,\mathrm{d}z = \lim_{\epsilon\to 0}\left(\int_{\gamma_\epsilon} H(z)\,\mathrm{d}z + \int_{\gamma_+} H(z)\,\mathrm{d}z\right) = 0.$$

As a result, H is holomorphic in $\mathbb{C}\setminus(-\infty,0]$, and hence $f(z) = h(z)$, $z \in \mathbb{D}\setminus(-1,0]$.

Step 3: Conclusion using the holomorphy of f. — We have proved that $f = h$. But $f \in H^2$, and as such, is holomorphic on $\mathbb{D}$, while the definition of h (26) involves functions with a cut along $\mathbb{R}_-$. The aim is to prove that all the coefficients in (26) are zero by using that for all $0 < r < 1$, $h(-r+\mathrm{i}0) = h(-r-\mathrm{i}0)$.

Assume that at least one of the $a_\ell^\pm \neq 0$ and set ℓ_0 the largest ℓ such that $(a_\ell^+, a_\ell^-) \neq (0,0)$. For $0 < r < 1$,

$$h(-r\pm\mathrm{i}0) = \lim_{\epsilon\to 0^+} h(-r\pm\mathrm{i}\epsilon) = \sum_{0\le\ell<k} (\ln(r)\pm\mathrm{i}\pi)^\ell\left(a_\ell^+\mathrm{e}^{\mathrm{i}\ln(r)\mp\pi} + a_\ell^-\mathrm{e}^{-\mathrm{i}\ln(r)\pm\pi}\right).$$

Setting $r = \mathrm{e}^s$ and taking $s\to-\infty$, we write this as

$$\begin{aligned} h(-\mathrm{e}^s\pm\mathrm{i}0) &= (s\pm\mathrm{i}\pi)^{\ell_0}\left(a_{\ell_0}^+\mathrm{e}^{\mathrm{i}s\mp\pi} + a_{\ell_0}^-\mathrm{e}^{-\mathrm{i}s+\pm\pi}\right) + O(s^{\ell_0-1})\\ &= s^{\ell_0}\left(a_{\ell_0}^+\mathrm{e}^{\mathrm{i}s\mp\pi} + a_{\ell_0}^-\mathrm{e}^{-\mathrm{i}s+\pm\pi}\right) + O(s^{\ell_0-1}). \end{aligned}$$

But $f = h$ is supposed to be holomorphic on $\mathbb{D}$. Hence, for $s < 0$, $h(-\mathrm{e}^s+\mathrm{i}0) = h(-\mathrm{e}^s-\mathrm{i}0)$. In particular,

$$\begin{aligned} 0 &= h(-\mathrm{e}^s+\mathrm{i}0) - h(-\mathrm{e}^s-\mathrm{i}0)\\ &= s^{\ell_0}2\sinh(\pi)\left(-a_{\ell_0}^+\mathrm{e}^{\mathrm{i}s} + a_{\ell_0}^-\mathrm{e}^{-\mathrm{i}s}\right) + O(s^{\ell_0-1}). \end{aligned}$$

This is only possible if

$$-a_{\ell_0}^+\mathrm{e}^{\mathrm{i}s} + a_{\ell_0}^-\mathrm{e}^{-\mathrm{i}s} \xrightarrow[s\to-\infty]{} 0.$$

This is in turn only possible if $a_{\ell_0}^+ = a_{\ell_0}^- = 0$, a contradiction with $(a_{\ell_0}^+, a_{\ell_0}^-) \neq (0,0)$.

By contradiction, all the $a_\ell^\pm$ are 0, hence $f = h = 0$. □

With this proposition, we can easily prove that $\mathcal{NC}_{H^2}(\omega)$ is dense:

Proof of theorem 1.8. Let $h \in (\mathcal{NC}_{H^2}(\omega)\cap W^{k,2}(\mathbb{T}))^\perp$ (orthogonal in $H^2\cap W^{k,2}(\mathbb{T})$). Every $g \in C_c^\infty(\omega)$ satisfies the hypothesis of our sufficient condition (theorem 1.7) so that $P_+g \in \mathcal{NC}_{H^2}(\omega)$, and hence

$$0 = \langle h, P_+g\rangle_{W^{k,2}(\mathbb{T})} = \langle h, g\rangle_{W^{k,2}(\mathbb{T})}.$$

Thus, according to theorem 2.6 $h = 0$. This proves that $\mathcal{NC}_{H^2}(\omega)^\perp = \{0\}$. □

Finally, let us prove the necessary and sufficient conditions for $W^{1/2,2}$ initial states.

Proof of theorem 1.9. Step 1: Direct implication. — Let $f_0 \in \mathcal{NC}_{H^2}(\omega) \cap W^{1/2,2}(\mathbb{T})$. According to the necessary condition of theorem 1.5, there exists $g \in L^2(\mathbb{T})$ such that $f_0 = P_+ g$, $\operatorname{supp} g \subset \omega^{\mathrm{cl}}$ and $\sum_{n<0} |n||\widehat{g}(n)|^2 < +\infty$. Since $f_0 = P_+ g \in W^{1/2,2}(\mathbb{T})$ by hypothesis, we also have $\sum_{n\geq 0} |n||\widehat{g}(n)|^2 < +\infty$, thus $g \in W^{1/2,2}(\mathbb{T})$.

Step 2: Reciprocal implication. — Let $g \in W^{1/2,2}(\mathbb{T})$ with $\operatorname{supp} g \subset \omega^{\mathrm{cl}}$. Of course, $P_+ g \in H^2 \cap W^{1/2,2}(\mathbb{T})$. Let us prove $P_+ g \in \mathcal{NC}_{H^2}(\omega)$.

By definition, $g \in W_{00}^{1/2,2}(\omega)$ [32, Definition 1.85]. Hence, $\int_{\mathbb{T}} \frac{|f(\zeta)|^2}{|(\zeta-\zeta_1)(\zeta-\zeta_2)|} |\mathrm{d}\zeta| < \infty$ (see, e.g., [32, Theorem 1.87] for the analog property for $W_{00}^{1/2,2}(\mathbb{R}_+)$, and use some partition of unity).

Thus, according to our sufficient condition (theorem 1.7), $P_+ g \in \mathcal{NC}_{H^2}(\omega)$. □

The reciprocal implication can also be shown without theorem 1.7 in the following way. Consider $\tilde{g} : \partial\Omega \to \mathbb{C}$ defined by $\tilde{g} = g$ on ω, $\tilde{g} = 0$ on $\partial\Omega \setminus \omega$. Since $g \in W_{00}^{1/2,2}(\omega)$, $\tilde{g} \in W^{1/2,2}(\partial\Omega)$. Hence, there exists $h \in W^{1,2}(\Omega)$ such that the trace of h on $\partial\Omega$ is $\tilde{g}$ [33, Theorem 18.40]. Then, we can solve the Dirichlet problem (in the sense of distributions) $-\Delta u = \Delta h$ on Ω, $u \in W_0^{1,2}(\Omega)$. Notice that $v = u + h$ is the solution of $\Delta v = 0$ on Ω, $v = \tilde{g}$ on $\partial\Omega$. Since $v \in W^{1,2}(\Omega)$ a use of Stokes formula will give the result (see section 2.4.1 for more details).

2.4 Characterization thanks to the Poisson kernel

We will now prove the necessary and sufficient condition of theorem 1.10. We start presenting the main idea before giving a rigorous proof.

2.4.1 Formal argument

The main idea of the proof is essentially the same as in the proof of theorem 2.5 and based on Stokes' formula (19). Now, instead of extending zg harmonically to $\mathbb{D}_{\mathrm{e}}$ we use harmonic extension to Ω. Let $g \in L^2(\mathbb{T})$ with $\operatorname{supp} g \subset \omega^{\mathrm{cl}}$. As in the rest of the article, we consider $\Omega = \Omega_T$ as defined in fig. 1. We denote by u the solution of (see section B and theorem B.3 for the proper definition):

$$\begin{cases} \Delta u(z) = 0, & z \in \Omega \\ u(z) = zg(z), & z \in \omega \\ u(z) = 0, & z \in \partial\Omega \setminus \omega. \end{cases}$$

Here and in the following, zg is just the function $z \mapsto zg(z)$. Let $p \in \mathbb{C}[X]$. Since P_+ is self adjoint on $L^2(\mathbb{T})$ and $p \in H^2$,

$$\begin{aligned}
\langle p, P_+ g\rangle_{H^2} &= \langle p, g\rangle_{H^2} \\
&= \int_\omega p(z)\overline{g(z)}|\mathrm{d}z| && (\operatorname{supp} g \subset \omega^{\mathrm{cl}}) \\
&= \int_\omega z p(z)\overline{zg(z)}|\mathrm{d}z| && (|z|^2 = 1 \text{ on } \omega) \\
&= -\mathrm{i} \int_\omega p(z)\overline{zg(z)}\,\mathrm{d}z && (\text{on } \omega,\ \mathrm{d}z = \mathrm{i}z|\mathrm{d}z|) \\
&= -\mathrm{i} \int_{\partial\Omega} p(z)\overline{u(z)}\,\mathrm{d}z && (\text{boundary values of } u).
\end{aligned}$$

Consider the differential form $\omega = p(z)\overline{u}(z)\,\mathrm{d}z$ (where u is now the harmonic extension to Ω instead of $G = P^e[zg]$ appearing in (19)), an application of Stokes formula gives formally:

$$\begin{aligned}\langle p, P_+g\rangle_{H^2} &= -\mathrm{i}\int_\Omega \partial_{\overline{z}}(p(z)\overline{u(z)})\,\mathrm{d}\overline{z}\wedge\mathrm{d}z \\ &= 2\int_\Omega p(z)\overline{\partial_z u}(z)\,\mathrm{d}A(z) \qquad\qquad (\partial_{\overline{z}}(p)=0).\end{aligned}$$

Hence, if we want to estimate the left-hand side by $\|p\|_{L^2(\Omega)}$, it should be necessary and sufficient to have $\partial_z u \in L^2(\Omega)$.

2.4.2 Rigorous proof

If we assume $\partial_z u \in L^2(\Omega)$, we can justify the computation above by approximating Ω by a smooth subdomain (see section B and theorem B.5). Hence, the sufficient part of theorem 1.10 is proved.

However, the proof of the necessary part of theorem 1.10 will require additional ideas. The observability inequality (theorem 1.2) allows us to define ψ_g by:

Definition 2.7. Let $g \in L^2(\mathbb{T})$ with $\operatorname{supp} g \subset \omega^{\mathrm{cl}}$. If $P_+g \in \mathcal{NC}_{H^2}(\omega)$, we denote by ψ_g the unique function in $A^2(\Omega)$ such that

$$\forall p \in \mathbb{C}[X],\ \langle p, P_+g\rangle_{H^2} = \int_\Omega p(z)\overline{\psi_g(z)}\,\mathrm{d}A(z).$$

If g is regular enough, we can explicitely identify ψ_g:

Proposition 2.8. *If $g \in C_c^\infty(\omega)$ with $P_+g \in \mathcal{NC}_{H^2}(\omega)$, then $\psi_g = 2\partial_z P^\Omega(zg)$.*

Proof. If $g \in C_c^\infty(\omega)$ and g is extended by 0 on $\partial\Omega\setminus\omega$, standard elliptic regularity methods tell us that $P^\Omega g \in W^{1,2}(\Omega)$ (see theorem B.4 for a proof relying on the Riemann map), and this is also true for $P^\Omega(zg)$ since zg has the same regularity as g. Hence, $\partial_z P^\Omega(zg) \in L^2(\Omega)$ and we can apply the Stokes formula of theorem B.5. As in section 2.4.1, we get

$$\begin{aligned}\langle p, P_+g\rangle_{H^2} &= \langle p, g\rangle_{H^2} \\ &= 2\int_\Omega p(z)\overline{\partial_z P^\Omega(zg)(z)}\,\mathrm{d}A(z).\end{aligned}$$

Finally, since $P^\Omega(zg)$ is harmonic and since Ω is simply connected, $P^\Omega(zg)$ can be written as a sum of a holomorphic and an anti-holomorphic function, say $P^\Omega(zg) = f_1 + f_2$. Since f_2 is anti-holomorphic, $\partial_z f_2 = 0$. Hence, $\partial_z P^\Omega(zg) = f_1'$ is holomorphic. Since we have already proved that $\partial_z P^\Omega(zg) \in L^2(\Omega)$, it follows that $\partial_z P^\Omega(zg) \in A^2(\Omega)$. By uniqueness, $\psi_g = 2\partial_z P^\Omega(zg)$. ☐

Now, the necessary part of theorem 1.10 will be an easy consequence of the following two propositions:

Proposition 2.9. *Let $g, g_n \in L^2(\mathbb{T})$ be supported in ω^{cl}. Assume that $g_n \xrightarrow[n\to\infty]{} g$ in $L^2(\omega)$. Then, for every $z \in \Omega$,*

$$\partial_z P^\Omega(zg_n)(z) \xrightarrow[n\to+\infty]{} \partial_z P^\Omega(zg)(z).$$

Proposition 2.10. *Let $g, g_n \in L^2(\mathbb{T})$ be supported in ω^{cl} and such that $P_+g_n \in \mathcal{NC}_{H^2}(\omega)$ and $P_+g \in \mathcal{NC}_{H^2}(\omega)$. Assume that $g_n \xrightarrow[n\to\infty]{} g$ in $L^2(\omega)$. Then, for every $z \in \Omega$,*

$$\psi_{g_n}(z) \xrightarrow[n\to+\infty]{} \psi_g(z).$$

Proof of theorem 2.9. The map $g \mapsto P^\Omega g$ is continuous from $L^2(\omega)$ to $h(\Omega)$ endowed with the topology of uniform convergence on compact subsets (theorem B.3). Since $P^\Omega g$ is harmonic, for every $z \in \Omega$ and every $r > 0$ such that $\overline{D(z,r)} \subset \Omega$, a standard estimate on harmonic functions [18, Chapter 2, Theorem 7] tells us:

$$|\partial_z P^\Omega g(z)| \leq C_r \|P^\Omega g\|_{L^1(D(z,r))} \leq C_r' \|P^\Omega g\|_{L^\infty(D(z,r))}.$$

This proves that for every $z \in \Omega$, the map $g \in L^2(\omega) \mapsto \partial_z P^\Omega g(z)$ is continuous. $\square$

Proof of theorem 2.10. Step 1: Representation of ψ_g. — We claim that for all $h \in L^2(\omega)$ such that $P_+ h \in \mathcal{NC}_{H^2}(\omega)$,

$$\psi_h(z) = \int_\omega h(\zeta)\overline{k_z^\Omega(\zeta)}|\mathrm{d}\zeta|, \tag{27}$$

where k_z^Ω is the Bergman kernel on Ω. Recall that k_z^Ω is the unique function in $A^2(\Omega)$ such that for all $f \in A^2(\Omega)$,

$$f(z) = \langle f, k_z^\Omega\rangle_{A^2(\Omega)}.$$

Observe that using the boundary behaviour of the conformal map $\mathbb{D} \to \Omega$, theorem B.2 allows to show that for all $z \in \Omega$, k_z^Ω extends continuously on Ω^{cl}. Hence, according to Mergelyan's theorem [45, Theorem 20.5], there exists a sequence of polynomials $(p_k)_k$ such that $p_k \to k_z^\Omega$ in $L^\infty(\Omega)$.

Thus

$$\begin{aligned}
\psi_h(z) &= \int_\Omega \psi_h(\zeta)\overline{k_z^\Omega(\zeta)}\,\mathrm{d}A(\zeta) && \text{(definition of the Bergman kernel)}\\
&= \lim_{k\to\infty}\int_\Omega \psi_h(\zeta)\overline{p_k(\zeta)}\,\mathrm{d}A(\zeta) && (p_k \xrightarrow[k\to\infty]{} k_z^\Omega)\\
&= \lim_{k\to\infty}\langle P_+h, p_k\rangle_{H^2} && \text{(definition 2.7 of } \psi_h)\\
&= \lim_{k\to\infty}\langle h, p_k\rangle_{L^2} && (P_+ \text{ selfadjoint and } P_+p_k = p_k)\\
&= \lim_{k\to\infty}\int_\omega h(\zeta)\overline{p_k(\zeta)}|\mathrm{d}\zeta| && (\operatorname{supp} h \subset \omega^{\mathrm{cl}})\\
&= \int_\omega h(\zeta)\overline{k_z^\Omega(\zeta)}|\mathrm{d}\zeta| && (p_k \xrightarrow[k\to\infty]{} k_z^\Omega).
\end{aligned}$$

This is the claimed formula.

Step 2: Conclusion. — We can apply this formula for g and g_n. Thus,

$$\begin{aligned}
\psi_g(z) &= \int_\omega g(\zeta)\overline{k_z^\Omega(\zeta)}|\mathrm{d}\zeta|\\
&= \lim_{n\to+\infty}\int_\omega g_n(\zeta)\overline{k_z^\Omega(\zeta)}|\mathrm{d}\zeta|\\
&= \lim_{n\to\infty}\psi_{g_n}(z).
\end{aligned}$$

$\square$

Combining these results, we can now prove the following formula for ψ_g:

Proposition 2.11. *Let $g \in L^2(\mathbb{T})$, be such that $\operatorname{supp} g \subset \omega^{\mathrm{cl}}$ and $P_+g \in \mathcal{NC}_{H^2}(\omega)$. Then,*

$$\psi_g = 2\partial_z P^\Omega(zg).$$

Since $\psi_g \in A^2(\Omega)$, this formula completes the proof of the necessary part of theorem 1.10.

Proof. Let $g_n \in C_c^\infty(\omega)$ be such that $g_n \to g$ in $L^2(\mathbb{T})$. According to theorem 1.9, $P_+ g_n \in \mathcal{NC}_{H^2}(\omega)$, and theorem 2.8 yields, $\psi_{g_n} = 2\partial_z P^\Omega(zg_n)$.

Thanks to theorem 2.9,

$$\psi_{g_n} = 2\partial_z P^\Omega(zg_n) \xrightarrow[n\to\infty]{} 2\partial_z P^\Omega(zg)$$

pointwise. Moreover, according to theorem 2.10,

$$\psi_{g_n} \xrightarrow[n\to\infty]{} \psi_g$$

pointwise. Thus, $\psi_g = 2\partial_z P^\Omega(zg)$. □

3 The L^2 problem

The aim of this section is to show how to use the H^2-results in order to prove the L^2-results.

3.1 The space $A^2 + \overline{A^2}$

We recall that for a bounded domain $\Omega \subset \mathbb{C}$, $A^2(\Omega)$ denotes the Bergman space on Ω, i.e., the set of holomorphic function on Ω that are in $L^2(\Omega)$.

To prove theorem 1.11, we need the following proposition:

Proposition 3.1. *Let Ω be an open bounded subset of $\mathbb{C}$. Assume that $\partial\Omega$ consists of a finite number of closed curves having a continuous tangent except at a finite number of corners of angle in $(0, 2\pi)$. Then, $A^2(\Omega) + \overline{A^2(\Omega)}$ is closed.*

Actually, it is known [44, Corollary 4.3] that $A^2(\Omega) + \overline{A^2(\Omega)} = \{f \in L^2(\Omega, \mathbb{C}),\ f \text{ is harmonic}\}$. As observed in [44, Section 4] the proposition above is a consequence of the following inequality of Friedrichs [20]: if Ω satisfies the condition of the proposition, then there is $0 < \theta < 1$ such that for every $f \in A^2(\Omega)$ with $\int_\Omega f \,\mathrm{d}A = 0$, we have

$$\left| \int_\Omega f(z)^2 \,\mathrm{d}A(z) \right| \leq \theta \int_\Omega |f(z)|^2 \,\mathrm{d}A(z).$$

See also [47], [43] for more on Friedrich's inequality and alternative proofs.

For convenience of the reader, let us deduce theorem 3.1 from the Friedrichs inequality, using some ingredients from [44, Corollary 4.4].

Proof of theorem 3.1 thanks to Friedrichs' inequality. Let $H := \{f \in A^2(\Omega),\ \int_\Omega f \,\mathrm{d}A = 0\}$. H is a closed subspace of $L^2(\Omega)$. Let $\Pi : L^2(\Omega) \to H$ be the orthogonal projection onto H, and let F be the antilinear operator $f \in H \mapsto \Pi\overline{f} \in H$.

Step 1: $\|F\| \leq \theta$, where θ is the constant of Friedrichs' inequality. — Thanks to the definition of F, we get that for every $f, g \in H$,

$$\langle f, Fg\rangle_{L^2} = \int_\Omega f(z)g(z) \,\mathrm{d}A(z).$$

In particular, $\langle f, Fg\rangle_{L^2} = \langle g, Ff\rangle_{L^2}$. Thus, thanks to the semilinearity of the scalar product,

$$\langle f, Fg\rangle_{L^2} = \frac{1}{4}(\langle f+g, F(f+g)\rangle_{L^2} - \langle f-g, F(f-g)\rangle_{L^2}).$$

Hence, according to Friedrichs' inequality and the parallelogram identity,

$$|\langle f, Fg\rangle_{L^2}| \leq \frac{1}{4}(\theta\|f+g\|^2 + \theta\|f-g\|^2) = \frac{\theta}{4}(2\|f\|^2 + 2\|g\|^2) = \frac{\theta}{2}(\|f\|^2 + \|g\|^2).$$

Finally, applying this to $f = \|g\|_{L^2}\|Fg\|_{L^2}^{-1}Fg$,

$$\|Fg\| \leq \theta\|g\|.$$

Step 2: Conclusion. — Let $f_n, g_n \in A^2(\Omega)$ be such that $h_n := f_n + \overline{g_n}$ converges in $L^2(\Omega)$, and let us prove that the limit h is in $A^2(\Omega) + \overline{A^2(\Omega)}$. Since $\lambda_n := \int_\Omega h_n \,\mathrm{d}A$ converges, we can assume without loss of generality that $\int_\Omega f_n \,\mathrm{d}A = \int_\Omega g_n \,\mathrm{d}A = 0$ (subtract their average otherwise).

According to Cauchy-Schwarz inequality and step 1, we have for every $f, g \in H$

$$\begin{aligned}
\|f + \overline{g}\|_{L^2}^2 &= \|f\|_{L^2}^2 + \|g\|_{L^2}^2 + 2\operatorname{Re}\int_\Omega f(z)g(z)\,\mathrm{d}A(z) \\
&= \|f\|_{L^2}^2 + \|g\|_{L^2}^2 + 2\operatorname{Re}\langle f, Fg\rangle_{L^2} \\
&\geq \|f\|_{L^2}^2 + \|g\|_{L^2}^2 - 2\theta\|f\|_{L^2}\|g\|_{L^2} \\
&\geq (1-\theta)(\|f\|_{L^2}^2 + \|g\|_{L^2}^2).
\end{aligned}$$

Thus, f_n and g_n are Cauchy sequences, hence they converge in $A^2(\Omega)$. This proves that $h = \lim(f_n + \overline{g_n}) = \lim f_n + \lim \overline{g_n} \in A^2(\Omega) + \overline{A^2(\Omega)}$. □

Theorem 3.1 allows us to prove the following result, where $L^2_0([0,T]\times\omega)$ is the space $\{f \in L^2([0,T]\times\omega),\ \int_{[0,T]\times\omega} f(t, \mathrm{e}^{\mathrm{i}x})\,\mathrm{d}t\,\mathrm{d}x = 0\}$. Recall the input-to-output map for the L^2 control system (2) defined on $L^2([0,T]\times\omega)$:

$$\mathcal{F}_T u := \int_0^T S(T-s)(\chi_\omega u(s,\cdot))\,\mathrm{d}s,$$

and correspondingly for eq. (3):

$$\mathcal{F}_T^+ = P_+\mathcal{F}_T.$$

Corollary 3.2. *Let $T > 0$ and $\mathcal{U} := \{v \in L^2([0,T]\times\omega),\ \mathcal{F}_T^+ v = 0\}$ the set of controls v that steer 0 to 0 in time T for the system $(\partial_t + |D|)h = P_+\mathbb{1}_\omega v$. We have*

$$\mathcal{U} + \overline{\mathcal{U}} = L^2_0([0,T]\times\omega).$$

Proof. We have $\mathcal{U} = \ker(P_+\mathcal{F}_T) = \operatorname{Range}((\mathcal{F}_T)^*P_+)^\perp$. As already mentioned in (10), $(\mathcal{F}_T)^*g(t) = \mathbb{1}_\omega^* S(T-t)^*g$, where $\mathbb{1}_\omega^* : L^2(\mathbb{T}) \to L^2(\omega)$ is the restriction operator. Hence, if $g_0 \in L^2(\mathbb{T})$ and $g_0^+ = P_+g_0 \in H^2$, $(\mathcal{F}_T)^*P_+g_0(t, \mathrm{e}^{\mathrm{i}x}) = g_0^+(\mathrm{e}^{-(T-t)+\mathrm{i}x})$ for $0 < t < T$ and $\mathrm{e}^{\mathrm{i}x} \in \omega$. Thus, setting $z = \mathrm{e}^{-(T-t)+\mathrm{i}x}$,

$$\begin{aligned}
u \in \mathcal{U} &\Leftrightarrow \forall g_0^+ \in H^2, \int_{[0,T]\times\omega} u(t, \mathrm{e}^{\mathrm{i}x})\overline{g_0^+(\mathrm{e}^{-(T-t)+\mathrm{i}x})}\,\mathrm{d}t\,\mathrm{d}x = 0 \\
&\Leftrightarrow \forall g_0^+ \in H^2, \int_{\Omega^*} |z|^{-2}u(T + \ln|z|, z/|z|)\overline{g_0^+(z)}\,\mathrm{d}A(z) = 0 \\
&\Leftrightarrow u_{\mathrm{a}} \in (H^2)^\perp \text{ (orthogonal in } L^2(\Omega^*)),
\end{aligned}$$

where $u_{\mathrm{a}}(z) = |z|^{-2}u(T + \ln|z|, z/|z|)$, and where as in the rest of the article, $\Omega^* = \Omega_T^* = \{z \in \mathbb{C},\ \mathrm{e}^{-T} < |z| < 1,\ z/|z| \in \omega\}$. Since polynomials are dense in H^2 and in $A^2(\Omega^*)$ [11, Chapter 18, Theorem 1.11], we get

$$u \in \mathcal{U} \Leftrightarrow u_{\mathrm{a}} \in (A^2(\Omega^*))^\perp.$$

Hence, using the same notation u_{a} as above,

$$u \in \mathcal{U} + \overline{\mathcal{U}} \Leftrightarrow u_{\mathrm{a}} \in A^2(\Omega^*)^\perp + \overline{A^2(\Omega^*)}^\perp.$$

Since $A^2(\Omega^*) + \overline{A^2(\Omega^*)}$ is closed (theorem 3.1), we get $A^2(\Omega^*)^\perp + \overline{A^2(\Omega^*)}^\perp = \left(A^2(\Omega^*) \cap \overline{A^2(\Omega^*)}\right)^\perp$ [8, Theorem 2.16]. Hence, $A^2(\Omega^*)^\perp + \overline{A^2(\Omega^*)}^\perp = \mathbb{C}^\perp$ (orthogonal of constant functions in $L^2(\Omega^*)$). Hence,

$$\begin{aligned} u \in \mathcal{U} + \overline{\mathcal{U}} &\Leftrightarrow u_{\mathrm{a}} \in \mathbb{C}^\perp \\ &\Leftrightarrow \int_{\Omega^*} |z|^{-2} u(T + \ln|z|, \arg(z/|z|))\, \mathrm{d}A(z) = 0 \\ &\Leftrightarrow \int_{[0,T]\times\omega} u(t, \mathrm{e}^{\mathrm{i}x})\, \mathrm{d}t\, \mathrm{d}x = 0. \end{aligned}$$

□

3.2 Link between the H^2 and L^2 control systems

Before proving theorem 1.11, let us point out that

$$P_+\overline{g}(\mathrm{e}^{\mathrm{i}x}) = P_+\left(\sum_{n\in\mathbb{Z}} \overline{\widehat{g}(n)}\mathrm{e}^{-\mathrm{i}nx}\right) = \sum_{n\leq 0} \overline{\widehat{g}(n)}\mathrm{e}^{-\mathrm{i}nx}.$$

Hence, as P_+g is the projection on nonnegative frequencies of g, $\overline{P_+\overline{g}}$ is the projection on nonpositive frequencies, a fact we will use frequently in the rest of this section.

Proof of theorem 1.11. Step 1: The easy implication. — If

$$(\partial_t + |D|)f(t, \mathrm{e}^{\mathrm{i}x}) = \mathbb{1}_\omega u(t, \mathrm{e}^{\mathrm{i}x}), \qquad f(0, \mathrm{e}^{\mathrm{i}x}) = f_0(\mathrm{e}^{\mathrm{i}x}),$$

then

$$\begin{aligned} (\partial_t + |D|)P_+f(t, \mathrm{e}^{\mathrm{i}x}) &= P_+\mathbb{1}_\omega u(t, \mathrm{e}^{\mathrm{i}x}), \qquad & P_+f(0, \mathrm{e}^{\mathrm{i}x}) &= P_+f_0(\mathrm{e}^{\mathrm{i}x}), \\ (\partial_t + |D|)P_+\overline{f(t, \mathrm{e}^{\mathrm{i}x})} &= P_+\mathbb{1}_\omega \overline{u(t, \mathrm{e}^{\mathrm{i}x})}, \qquad & P_+\overline{f(0, \mathrm{e}^{\mathrm{i}x})} &= P_+\overline{f_0}(\mathrm{e}^{\mathrm{i}x}). \end{aligned}$$

Hence, if $f_0 \in \mathcal{NC}_{L^2}(\omega, T)$, then $P^+f_0 \in \mathcal{NC}_{H^2}(\omega)$ and $P^+\overline{f_0} \in \mathcal{NC}_{H^2}(\omega)$.

Step 2: The less easy implication. — Assume that $P_+f_0 \in \mathcal{NC}_{H^2}(\omega)$ and $P_+\overline{f_0} \in \mathcal{NC}_{H^2}(\omega)$. Then there exists $u^\pm \in L^2([0,T] \times \omega)$ such that the solutions $f^\pm$ of

$$\begin{aligned} (\partial_t + |D|)f^+(t, \mathrm{e}^{\mathrm{i}x}) &= P_+\mathbb{1}_\omega u^+(t, \mathrm{e}^{\mathrm{i}x}), \qquad & f^+(0, \mathrm{e}^{\mathrm{i}x}) &= P_+f_0(\mathrm{e}^{\mathrm{i}x}), \\ (\partial_t + |D|)f^-(t, \mathrm{e}^{\mathrm{i}x}) &= P_+\mathbb{1}_\omega \overline{u^-(t, \mathrm{e}^{\mathrm{i}x})}, \qquad & f^-(0, \mathrm{e}^{\mathrm{i}x}) &= P_+\overline{f_0}(\mathrm{e}^{\mathrm{i}x}). \end{aligned} \tag{28}$$

satisfy $f^\pm(T, \cdot) = 0$.

We are looking for a v that steers f_0 to 0 for the L^2-control system (2). Let us denote by $f = S(t)f_0 + \mathcal{F}_t v$ the corresponding solution. Considering P_+f and $\overline{P_+\overline{f}}$, we see that v steers f_0 to 0 if and only if v steers both

- P_+f_0 (the nonnegative frequencies of f_0) to 0 for the control system $(\partial_t + |D|)h^+ = P_+\mathbb{1}_\omega v$,
- $\overline{P_+\overline{f_0}}$ (i.e., the nonpositive frequencies of f_0) to 0 for the control system $(\partial_t + |D|)h^- = \overline{P_+\mathbb{1}_\omega\overline{v}}$.

The first of the above two conditions is equivalent to $v \in u^+ + \mathcal{U}$, while the second condition is equivalent to $v \in u^- + \overline{\mathcal{U}}$. Hence, we want to prove that $(u^+ + \mathcal{U}) \cap (u^- + \overline{\mathcal{U}}) \neq \emptyset$. This is equivalent to $u^+ - u^- \in \mathcal{U} + \overline{\mathcal{U}}$.

But $\mathcal{U} + \overline{\mathcal{U}} = L^2_0([0,T] \times \omega)$ (theorem 3.2). Hence, we only have to prove that

$$\int_{[0,T]\times\omega} u^+(t, \mathrm{e}^{\mathrm{i}x})\,\mathrm{d}t\,\mathrm{d}x = \int_{[0,T]\times\omega} u^-(t, \mathrm{e}^{\mathrm{i}x})\,\mathrm{d}t\,\mathrm{d}x. \tag{29}$$

Considering the 0th Fourier coefficient in the control systems (28), we see that

$$\partial_t \widehat{f^+}(t,0) = \widehat{\mathbb{1}_\omega u^+}(t,0).$$

Hence,

$$\int_{[0,T]\times\omega} u^+(t, \mathrm{e}^{\mathrm{i}x})\,\mathrm{d}t\,\mathrm{d}x = \int_0^T \widehat{\mathbb{1}_\omega u^+}(t,0)\,\mathrm{d}t = \widehat{f^+}(T,0) - \widehat{f^+}(0,0) = -\widehat{f_0}(0).$$

Similarly,

$$\int_{[0,T]\times\omega} \overline{u^-}(t, \mathrm{e}^{\mathrm{i}x})\,\mathrm{d}t\,\mathrm{d}x = -\widehat{\overline{f_0}}(0).$$

Thus, we do have eq. (29), and the proof of theorem 1.11 is complete. □

3.3 Properties of the L^2 control system

We now combine the properties of the H^2 control system and the link between the H^2 and L^2 control system (theorem 1.11).

Proof of theorem 1.13. Assume that $f_0 \in \mathcal{NC}_{L^2}(\omega)$ is analytic in a neighborhood of $\mathbb{T}$, in other words, for some $c > 0$,

$$|\widehat{f_0}(n)| \leq C\mathrm{e}^{-c|n|}.$$

In particular, $P_+ f_0(z) = \sum_{n\geq 0} \widehat{f_0}(n) z^n$ extends holomorphically to $D(0, \mathrm{e}^c)$. But $P_+ f_0 \in \mathcal{NC}_{H^2}(\omega)$ (theorem 1.11). Hence, according to theorem 1.6 $P_+ f_0 = 0$.

Similarly, $P_+ \overline{f_0} = \sum_{n\leq 0} \overline{\widehat{f_0}(n)} z^{-n}$ is in $\mathcal{NC}_{H^2}(\omega)$ and extends analytically to $D(0, \mathrm{e}^c)$, hence $P_+ \overline{f_0} = 0$.

Thus, Fourier coefficients for both nonnegative and nonpositive n are 0, and so $f_0 = 0$. □

To prove that $\mathcal{NC}_{L^2}$ is dense, we will often use that P^+ is self-adjoint on $W^{k,2}(\mathbb{T})$. We will also use the following variation of theorem 2.6:

Lemma 3.3. *Let $k \in \mathbb{N}$. Let $f \in H^2 \cap W^{k,2}(\mathbb{T})$. Assume that*

$$\forall g \in C_c^\infty(\omega) \text{ such that } \int_\omega g = 0,\ \langle f, g \rangle_{W^{k,2}(\mathbb{T})} = 0.$$

Then f is constant.

Proof. Choose $\chi \in C_c^\infty(\omega)$ real valued with $\int_\omega \chi = 1$. Set $f_1 = f - \langle f, \chi \rangle_{W^{k,2}}$.

Let $g \in C_c^\infty(\omega)$. Set $g_1 = g - \chi \int_\omega g$, which satisfies $g_1 \in C_c^\infty(\omega)$ and $\int_\omega g_1 = 0$.

$$\begin{aligned}
\langle f_1, g \rangle_{W^{k,2}} &= \langle f, g \rangle_{W^{k,2}} - \langle f, \chi \rangle_{W^{k,2}} \langle 1, g \rangle_{W^{k,2}} \\
&= \langle f, g \rangle_{W^{k,2}} - \langle f, \langle g, 1 \rangle_{W^{k,2}} \chi \rangle_{W^{k,2}} \\
&= \left\langle f, g - \langle g, 1 \rangle_{W^{k,2}} \chi \right\rangle_{W^{k,2}}.
\end{aligned}$$

Notice that according to the definition of the $W^{k,2}$ scalar product, $\fint_\omega g = \langle g, 1\rangle_{W^{k,2}}$. Thus, according to the hypothesis on f,

$$\begin{aligned}\langle f_1, g\rangle_{W^{k,2}} &= \langle f, g_1\rangle_{W^{k,2}}\\ &= 0.\end{aligned}$$

Since this is true for every $g \in C_c^\infty(\omega)$ and since $f_1 \in H^2$, we can use theorem 2.6 to conclude that $f_1 = 0$, i.e., $f = \langle f, \chi\rangle_{W^{k,2}}$ is constant. ☐

Proof of theorem 1.14. Let $h \in \mathcal{NC}_{L^2}(\omega)^\perp$ (orthogonal in $W^{k,2}(\mathbb{T})$).

Step 1: P_+h is constant. — Let $g \in C_c^\infty(\omega)$ with $\fint_\mathbb{T} g = 0$ and set $g^+ = P_+g$. According to theorem 1.7, $P_+g^+ = P_+g \in \mathcal{NC}_{H^2}(\omega)$. Moreover, $P_+\overline{g^+}(z) = \widehat{\overline{g}}(0)$ ($P_+\overline{g^+}$ only sees the nonpositive frequencies of g^+, but $g^+ = P_+g$ does not have any negative frequencies). But we assumed that $\fint_\mathbb{T} g = 0$, hence $P_+\overline{g^+} = 0 \in \mathcal{NC}_{H^2}(\omega)$. Thus, $g^+ \in \mathcal{NC}_{L^2}(\omega)$ (theorem 1.11).

Thus, for every $g \in C_c^\infty(\omega)$ with $\fint_\mathbb{T} g = 0$,

$$0 = \langle h, g^+\rangle_{W^{k,2}(\mathbb{T})} = \langle h, P_+g\rangle_{W^{k,2}(\mathbb{T})} = \langle P_+h, g\rangle_{W^{k,2}(\mathbb{T})}$$

In addition, since $h \in L^2(\mathbb{T})$, $P_+h \in H^2$. Thus, according to theorem 3.3, P_+h is constant.

Step 2: $\overline{P_+\overline{h}}$ is constant. — The reasoning is similar. Let $g \in C_c^\infty(\omega)$ with $\fint_\mathbb{T} g = 0$, and set $g^- = \overline{P_+\overline{g}}$. According to theorem 1.7, $P_+\overline{g^-} = P_+\overline{g} \in \mathcal{NC}_{H^2}(\omega)$. Moreover, $P_+g^-(z) = \widehat{g}(0) = 0 \in \mathcal{NC}_{H^2}(\omega)$. Thus, $g^- \in \mathcal{NC}_{L^2}(\omega)$ (theorem 1.11).

Thus, for every $g \in C_c^\infty(\omega)$ with $\fint_\mathbb{T} g = 0$,

$$0 = \langle h, g^-\rangle_{W^{k,2}(\mathbb{T})} = \langle h, \overline{P_+\overline{g}}\rangle_{W^{k,2}(\mathbb{T})} = \overline{\langle \overline{h}, P_+\overline{g}\rangle_{W^{k,2}(\mathbb{T})}} = \overline{\langle P_+\overline{h}, \overline{g}\rangle_{W^{k,2}(\mathbb{T})}} = \langle \overline{P_+\overline{h}}, g\rangle_{W^{k,2}(\mathbb{T})}.$$

Thus $\overline{P_+\overline{h}}$ is constant.

Step 3: Conclusion. — From the two first steps, every positive frequency of h is 0, and from the second step, every negative frequency is 0, so that h is constant.

Finally, consider $g \in C_c^\infty(\omega)$ with $\fint_\mathbb{T} g \neq 0$. According to theorem 1.7, $P_+g \in \mathcal{NC}_{H^2}(\omega)$ and $P_+\overline{g} \in \mathcal{NC}_{H^2}(\omega)$. Hence, according to theorem 1.11, $g \in \mathcal{NC}_{L^2}(\omega)$. Consequently, $0 = \langle h, g\rangle_{W^{k,2}(\mathbb{T})}$, and since h is constant, $0 = \langle h, g\rangle_{W^{k,2}(\mathbb{T})} = \langle h, g\rangle_{L^2} = \widehat{h}(0)\fint_\mathbb{T} g$, so that finally $h = 0$.

We proved that $\mathcal{NC}_{L^2}(\omega)^\perp = \{0\}$. ☐

A Construction of the cutoff function

Lemma A.1. *Let Ω_1 and Ω_2 be open subsets of $\mathbb{C}$ such that $\mathrm{d}(\Omega_1 \setminus \Omega_2\ \Omega_2 \setminus \Omega_1) > 0$. There exists a function $\chi \in C^\infty(\mathbb{C})$ such that $\chi = 0$ on a neighborhood of $(\Omega_1 \setminus \Omega_2)^{\mathrm{cl}}$ and $\chi = 1$ on a neighborhood of $(\Omega_2 \setminus \Omega_1)^{\mathrm{cl}}$, with additionally $0 \leq \chi \leq 1$ and*

$$\forall x \in \Omega_1 \cup \Omega_2,\ |\nabla\chi(x)| \leq \frac{C}{\mathrm{d}(\Omega_1 \setminus \Omega_2,\ \Omega_2 \setminus \Omega_1)} \tag{30}$$

on $\Omega_1 \cap \Omega_2$.

Proof. Set $\epsilon = \mathrm{d}(\Omega_1 \setminus \Omega_2,\ \Omega_2 \setminus \Omega_1)$. Let $\rho \in C_c^\infty(B(0,1))$ with $\rho \geq 0$ and $\int_{\mathbb{C}} \rho(z)\,\mathrm{d}A(z) = 1$. For $\eta > 0$, set $\rho_\eta(z) = \eta^{-2}\rho(\eta^{-1}z)$.

Set $d_1(z) = \mathrm{d}(z,\ \Omega_1 \setminus \Omega_2)$ and $d_2(z) = \mathrm{d}(z,\ \Omega_2 \setminus \Omega_1)$. Set $\delta_i^\eta = d_i * \rho_\eta$.

Let $\varphi \in C^\infty([0,1])$ with $\varphi = 0$ on $[0,1/3]$ and $\varphi = 1$ on $[2/3,1]$.

Finally, for some $\eta > 0$ to be chosen later (depending on ϵ), set

$$\chi(z) = \varphi\left(\frac{\delta_1^\eta(z)}{\delta_1^\eta(z) + \delta_2^\eta(z)}\right). \tag{31}$$

First, we claim that

$$\|d_i - \delta_i^\eta\|_{L^\infty} \leq C_1\eta. \tag{32}$$

This a consequence of straightforward computations using that d_i is 1-lipschitz:

$$\begin{aligned}
|\delta_i^\eta(z) - d_i(z)| &\leq \int_{\mathbb{C}} |d_i(w) - d_i(z)|\rho_\eta(z-w)\,\mathrm{d}A(w) \\
&\leq \int_{\mathbb{C}} |z-w|\rho_\eta(z-w)\,\mathrm{d}A(w) \\
&= \eta \int_{\mathbb{C}} |w|\rho(w)\,\mathrm{d}A(w).
\end{aligned}$$

From now on, we choose $\eta = \epsilon/(16C_1)$ (with the C_1 of the previous estimate (32)), so that $\|d_i - \delta_i^\eta\|_{L^\infty} \leq \epsilon/16$.

Here are some facts about these functions:

1. $|\nabla d_i| \leq 1$, hence $\|\nabla \delta_i^\eta\|_{L^\infty} = \|\nabla d_i * \rho_\eta\|_{L^\infty} \leq \|\nabla d_i\|_{L^\infty}\|\rho_\eta\|_{L^1} \leq 1$.
2. By definition of ϵ, we cannot have both $d_1 < \epsilon/2$ and $d_2 < \epsilon/2$, hence $d_1 + d_2 \geq \epsilon/2$, and finally $\delta_1^\eta + \delta_2^\eta \geq d_1 + d_2 - \epsilon/8 \geq 3\epsilon/8$.
3. If $d_1 \leq \epsilon/16$, we have $\delta_1^\eta \leq \epsilon/16 + \epsilon/16$, hence $\delta_1^\eta/(\delta_1^\eta + \delta_2^\eta) \leq \frac{\epsilon/8}{3\epsilon/8} = 1/3$. Hence, by definition of χ (eq. (31)), $\chi = 0$ on a neighborhood of $(\Omega_1 \setminus \Omega_2)^{\mathrm{cl}}$.
4. Similarly, $\chi = 1$ on $\{z,\ d_2(z) < \epsilon/16\}$, which is a neighborhood of $(\Omega_2 \setminus \Omega_1)^{\mathrm{cl}}$.

The only thing left to prove on χ is the gradient estimate (14). We have

$$\begin{aligned}
\partial_x \chi &= \partial_x\left(\frac{\delta_1^\eta}{\delta_1^\eta + \delta_2^\eta}\right)\varphi'\left(\frac{\delta_1^\eta}{\delta_1^\eta + \delta_2^\eta}\right) \\
&= \frac{\delta_2^\eta \partial_x \delta_1^\eta - \delta_1^\eta \partial_x \delta_2^\eta}{(\delta_1^\eta + \delta_2^\eta)^2}\varphi'\left(\frac{\delta_1^\eta}{\delta_1^\eta + \delta_2^\eta}\right) \\
&= \left(\frac{\partial_x \delta_1^\eta}{1 + \delta_1^\eta/\delta_2^\eta} - \frac{\partial_x \delta_2^\eta}{1 + \delta_2^\eta/\delta_1^\eta}\right)\frac{1}{\delta_1^\eta + \delta_2^\eta}\varphi'\left(\frac{\delta_1^\eta}{\delta_1^\eta + \delta_2^\eta}\right).
\end{aligned}$$

In $\Omega_1 \cap \Omega_2$, $\delta_i^\eta > 0$, hence $1 + \delta_1^\eta/\delta_2^\eta \geq 1$. Hence, we can estimate the right-hand side as

$$|\partial_x \chi| \leq \left(|\partial_x \delta_1^\eta| + |\partial_x \delta_1^\eta|\right)\frac{1}{|\delta_1^\eta + \delta_2^\eta|}\|\varphi'\|_{L^\infty}.$$

Using that $|\nabla \delta_i^\eta| \leq 1$ and $\delta_1^\eta + \delta_2^\eta \geq 3\epsilon/8$, we get

$$|\partial_x \chi| \leq \frac{16}{3\epsilon}\|\varphi'\|_{L^\infty}.$$

The same holds for $\partial_y \chi$. $\square$

B Background on the boundary behaviour of conformal maps and on the Poisson kernel

In this section, $\Omega \subset \mathbb{C}$ satisfies:

$$\begin{array}{l}\Omega \text{ is a Jordan domain, i.e., the bounded component of } \mathbb{C} \setminus \gamma, \text{ where } \gamma \text{ is a closed Jordan curve,} \\ \partial\Omega \text{ is a finite union of smooth (say } C^2\text{) curves that meet at angle } \pi\alpha_k \text{ with } 0 < \alpha_k < 1.\end{array} \tag{H}$$

The cases we are interested in are $\Omega = \{e^x\zeta,\ 0 < x < T,\ \zeta \in \omega\} = \Omega_T$. Let also φ be a Riemann mapping from $\mathbb{D}$ to Ω.

Based on some existing results about the boundary behaviour of φ, we state and prove the results we need about the Poisson kernel and the Bergman Kernel of Ω.

We do not claim much novelty here: some of these results (or similar ones) are either in the literature in great generality or are probably folklore. But to help the reader, we still prove them in our specific case. Let us also mention that not every theorem that follows require all the above hypotheses on Ω, even with our proof strategy. Since our goal is not to get the most general statements possible, we do not detail this.

According to Carathéodory's theorem [42, Theorem 2.6], φ extends by continuity to $\mathbb{D}^{\mathrm{cl}}$, and this extension (still denoted φ) is a homeomorphism from $\mathbb{D}^{\mathrm{cl}}$ to Ω^{cl}. In fact, if $\zeta_k = \varphi(z_k) \in \partial\Omega$ is a corner of angle $\pi\alpha_k$, we have the following asymptotics when $z \to z_k$ [42, Theorem 3.9]:

$$\varphi(z) - \varphi(z_k) \sim c(z - z_k)^{\alpha_k}, \tag{33}$$

$$\varphi'(z) \sim c'(z - z_k)^{\alpha_k - 1}. \tag{34}$$

Note that this applies also when $\partial\Omega$ is smooth at ζ_0: this is the case when $\alpha = 1$.

Finally, let us mention that the linear measure $|\mathrm{d}z|$ on $\partial\Omega$ is given by $|\mathrm{d}z| = |\varphi'(w)||\mathrm{d}w|$ [42, Theorem 6.8], i.e., if $J \subset \partial\mathbb{D}$,

$$\int_{\varphi(J)} |\mathrm{d}z| = \int_J |\varphi'(w)||\mathrm{d}w|. \tag{35}$$

Here are some useful properties on φ that we will use in this appendix:

Lemma B.1. *Assume $\Omega \subset \mathbb{C}$ satisfies the hypothesis* (H)*.*

- *Let $\psi = \varphi^{-1}$. Then, ψ' extends continuously to Ω^{cl}. In particular, $|\varphi'| \geq c$ for some $c > 0$.*
- *There exists $C > 0$ such that for every $0 < r < 1$ and every $w \in \mathbb{T} \setminus \{\zeta_1, \dots, \zeta_n\}$, $|\varphi'(rw)| \leq C|\varphi'(w)|$.*

Proof. Step 1: First assertion. — Of course, ψ' is continuous and does not vanish inside Ω. To prove the first point, we only need to look on a neighborhood of each point of the boundary. Let $\zeta_0 = \varphi(z_0) \in \partial\Omega$. Then, inverting the asymptotics (33), we get

$$\psi(\zeta) - \psi(\zeta_0) \sim c^{-1/\alpha}(\zeta - \zeta_0)^{1/\alpha}.$$

Plugging this asymptotic in the asymptotics of φ', we get

$$\begin{aligned}\psi'(\zeta) &= \frac{1}{\varphi'(\psi(\zeta))} \\ &\sim \frac{1}{c'\left(c^{-1/\alpha}(\zeta - \zeta_0)^{1/\alpha}\right)^{\alpha - 1}} \\ &\sim c''(\zeta - \zeta_0)^{1/\alpha - 1}.\end{aligned}$$

Since $\alpha < 1$ if ζ_0 is a corner and $\alpha = 1$ else, ψ' has a limit in ζ_0 (equal to zero at the corners). Since this is true for any $\zeta_0 \in \partial\Omega$, ψ' does extend continuously to Ω^{cl}. Since $\psi' = 1/\varphi' \circ \psi$ extends continuously (and hence is bounded), $|\varphi'|$ is bounded from below.

Step 2: Second assertion. — According to the previous asymptotics (33)–(34), the functions:

$$\frac{\varphi(z) - \varphi(z_0)}{(z - z_0)^{\alpha}} \quad \text{and} \quad \frac{\varphi'(z)}{(z - z_0)^{\alpha-1}}$$

are continuous and different from 0 and ∞ in $\overline{\mathbb{D}} \cap D(z_0, \epsilon)$ for some $\epsilon > 0$ (this is even true when there is no corner, i.e. $\alpha = 1$). If $\{\zeta_k\}_{k=1}^n = \{\varphi(z_k)\}_{k=1}^n$ are the n corners of Ω with corresponding angles $\pi\alpha_k$, it follows that $h(z) = \varphi'(z)/\prod_{k=1}^n (z - z_k)^{\alpha_k - 1}$ extends to a non-vanishing continuous bounded function on $\overline{\mathbb{D}}$. In particular

$$|\varphi'(z)| \asymp \prod_{k=1}^{n} |z - z_k|^{\alpha_k - 1}$$

Let $|w| = 1$. It suffices to estimate $|rw - z_k|$. We have:

$$|w - z_k| \leq |w - rw| + |rw - z_k| \leq 2|rw - z_k|.$$

Since $0 < \alpha_k \leq 1$, this proves

$$|rw - z_k|^{\alpha_k - 1} \leq 2^{1-\alpha}|w - z_k|^{\alpha_k - 1}.$$

□

As a first consequence, here is a property on the Bergman kernel that is used in the proof of our necessary and sufficient condition (theorem 1.10)

Proposition B.2. *Assume $\Omega \subset \mathbb{C}$ satisfies the hypothesis (H). Let $w \in \Omega$ and let k_w^{Ω} be the Bergman kernel (i.e., for all $f \in A^2(\Omega)$, $f(w) = \langle f, k_w^{\Omega}\rangle_{A^2(\Omega)}$). Then, for all $w \in \Omega$, k_w^{Ω} extends continuously on $\overline{\Omega}$.*

Proof. Let $k_w^{\mathbb{D}}(z) = (\pi(1 - z\overline{w})^2)^{-1}$ be the Bergman kernel of the unit disk [50, Theorem 2.7, 16, §1.2]. Set $\psi = \varphi^{-1}$. By the change of variables formula (17),

$$k_w^{\Omega}(z) = k_{\psi(w)}^{\mathbb{D}}(\psi(z))\psi'(z)\overline{\psi'(w)}.$$

According to the formula for $k^{\mathbb{D}}$, for every $w \in \mathbb{D}$, $k_w^{\mathbb{D}}$ is continuous on $\mathbb{D}^{\mathrm{cl}}$. Since ψ is continuous on $\mathbb{D}^{\mathrm{cl}}$ (Carathéordory's theorem) and ψ' is also continuous on $\mathbb{D}^{\mathrm{cl}}$ (theorem B.1), k_w^{Ω} is indeed continuous on $\mathbb{D}^{\mathrm{cl}}$. □

Another consequence is that the Dirichlet problem is solvable on Ω with L^2 boundary data. Recall that a function h is harmonic in Ω if and only if $h \circ \varphi$ is harmonic in $\mathbb{D}$. In particular, if h is regular enough (say $C^2(\Omega) \cap C^0(\overline{\Omega})$), $g := h_{|\partial\Omega}$ and $\Delta h = 0$, then, $\tilde{h} = h \circ \varphi$ satisfies $\tilde{h} \in C^2(\mathbb{D}) \cap C^0(\overline{\mathbb{D}})$ (assuming hypothesis (H)), $\Delta\tilde{h} = 0$ and $\tilde{h}_{|\partial\mathbb{D}} = h \circ \varphi$. I.e., $h \circ \varphi = P^{\mathbb{D}}(g \circ \varphi)$. This motivates the following definition of P^{Ω}:

Corollary B.3. *Assume $\Omega \subset \mathbb{C}$ satisfies the hypothesis (H). Let P^{Ω} be the map $C^0(\partial\Omega) \to C^0(\Omega^{\mathrm{cl}})$ defined by*

$$P^{\Omega}g(\varphi(w)) = P^{\mathbb{D}}(g \circ \varphi)(w), \quad w \in \mathbb{D},$$

where $P^{\mathbb{D}}$ is the Poisson kernel of the disk:

$$P^{\mathbb{D}}(z, \zeta) = \frac{1 - |z|^2}{|z - \zeta|^2}, \quad z \in \mathbb{D}, \zeta \in \mathbb{T}.$$

The formula above extends P^{Ω} as a continuous linear map $L^2(\partial\Omega) \to h(\Omega)$, where $h(\Omega)$ is the space of harmonic functions on Ω endowed with the topology of uniform convergence on compact subsets.

We have already encountered the Poisson kernel for domains other than $\mathbb{D}$ when discussing harmonic extension to $\mathbb{D}_e$, but this domain does not satisfy the hypotheses (H).

By definition, we say that $u = P^\Omega g$ is the solution of $\Delta u = 0$ on Ω, $u = g$ on $\partial\Omega$. The Dirichlet problem in nonsmooth domains has been widely studied, with hypotheses much more general than (H), see, e.g., [27] for an overview. But in our case, we can prove what we need very easily.

Proof. As in theorem B.1, we denote $\psi = \varphi^{-1}$. The definition of P^Ω reads

$$P^\Omega g(\varphi(w)) = \int_{\partial\mathbb{D}} P^{\mathbb{D}}(w, w')g(\varphi(w'))|\mathrm{d}w'|, \quad w \in \mathbb{D}.$$

According to the change of variable-formula for $|\mathrm{d}w'|$ (35), this can be written as

$$P^\Omega g(z) = \int_{\partial\Omega} P^{\mathbb{D}}(\psi(z), \psi(z'))g(z')|\psi'(z')||\mathrm{d}z'|, \quad z = \varphi(w) \in \Omega.$$

Since $|\psi'|$ is bounded (theorem B.1), the definition of $P^\Omega g$ makes sense for $g \in L^2(\partial\Omega)$, and for every such g, $P^\Omega g$ is harmonic on Ω. The required continuity property of the map $g \mapsto P^\Omega g$ follows from the fact that $P^{\mathbb{D}}(\psi(z), \psi(z')|\psi'(z')|$ is bounded on $D(0,r)_z \times \partial\mathbb{D}_{z'}$ for every $0 < r < 1$. □

The following proposition is a simple elliptic regularity result. Since it is enough for our purpose and since the proof is simple with the properties of the conformal map outlined above, we do it here.

Proposition B.4. *Assume $\Omega \subset \mathbb{C}$ satisfies the hypothesis (H). Let $g : \partial\Omega \to \mathbb{C}$. For every $\zeta_0 \in \partial\Omega$ such that $\partial\Omega$ is not smooth at ζ_0, assume that $g = 0$ on a neighborhood of ζ_0. If $g \in W^{1,2}(\partial\Omega)$, then $P^\Omega g \in W^{1,2}(\Omega)$.*

Proof. Let $\zeta_0 = \varphi(z_0) \in \partial\Omega$ that is not a corner of $\partial\Omega$. According to the asymptotics of φ and φ' (33)–(34), φ is C^1 at z_0.

Since $g = 0$ on a neighborhood of the corners, we deduce that $g \circ \varphi \in W^{1,2}(\mathbb{T})$. We claim that $\partial_z P^{\mathbb{D}}(g \circ \varphi) \in L^2(\mathbb{D})$. This is a standard computation: set $g_1 = g \circ \varphi$. We can write

$$P^{\mathbb{D}} g_1(z) = \sum_{n \geq 0} \widehat{g_1}(n) z^n + \sum_{n > 0} \widehat{g_1}(-n)\bar{z}^n.$$

In particular,

$$\|\partial_z(P^{\mathbb{D}} g_1)\|^2_{L^2(\mathbb{D})} = \left\| \sum_{n \geq 0} n\widehat{g_1}(n) z^{n-1} \right\|^2_{L^2(\mathbb{D})} = \pi \sum_{n > 0} n|\widehat{g_1}(n)|^2.$$

Since $g_1 \in W^{1,2}(\mathbb{T})$, the right-hand side above is indeed finite (in fact, we only need $g_1 \in W^{1/2,2}(\mathbb{T})$). This proves the claim.

Finally, the change of variables $z = \psi(\zeta) = \varphi^{-1}(\zeta)$ that satisfies $\mathrm{d}A(z) = |\psi'(\zeta)|^2 \,\mathrm{d}A(\zeta)$ gives

$$\begin{aligned}
\int_\Omega |\partial_\zeta P^\Omega g(\zeta)|^2 \,\mathrm{d}A(\zeta) &= \int_\Omega \left| \frac{\mathrm{d}}{\mathrm{d}\zeta} \left(P^{\mathbb{D}} g_1(\psi(\zeta))\right) \right|^2 \mathrm{d}A(\zeta) \\
&= \int_\Omega \left|\psi'(\zeta) \left(\partial_z P^{\mathbb{D}} g_1(\psi(\zeta))\right)\right|^2 \mathrm{d}A(\zeta) \\
&= \int_{\mathbb{D}} \left|\partial_z P^{\mathbb{D}} g_1(z)\right|^2 \mathrm{d}A(z).
\end{aligned}$$

This right-hand side is finite, hence $\partial_\zeta P^\Omega g \in L^2(\Omega)$. Similarly, $\partial_{\bar\zeta} P^\Omega g \in L^2(\Omega)$. □

Proposition B.5. *Assume $\Omega \subset \mathbb{C}$ satisfies the hypothesis* (H). *We use the notation P^Ω of theorem B.3. Let $g \in L^2(\partial\Omega)$. Assume that $\partial_z P^\Omega g \in L^2(\Omega)$. Then, for every v holomorphic on Ω that is $C^0(\Omega^{\mathrm{cl}})$, we have*

$$\int_{\partial\Omega} v(z)\overline{g}(z)\,\mathrm{d}z = 2\mathrm{i}\int_\Omega v(z)\overline{\partial_z P^\Omega g}(z)\,\mathrm{d}A(z).$$

If Ω is smooth and $f \in C^1(\Omega^{\mathrm{cl}})$, Stokes' theorem applied to the differential form $f(z)\,\mathrm{d}z$, gives

$$\int_{\partial\Omega} f(z)\,\mathrm{d}z = \int_\Omega \partial_{\overline{z}} f(z)\,\mathrm{d}\overline{z}\wedge\mathrm{d}z = 2\mathrm{i}\int_\Omega \partial_{\overline{z}} f(z)\,\mathrm{d}A(z).$$

If $v\overline{P^\Omega g} \in C^1(\Omega^{\mathrm{cl}})$, theorem B.5 is just this formula with $f = v\overline{P^\Omega g}$. The only difference in the statement above is the weaker regularity on Ω and g.

Proof. We denote $u = P^\Omega g$. For $0 < r < 1$, let $\Omega_r = \varphi(D(0,r))$ that is a smooth domain. We apply the usual Stokes formula and take the limit $r \to 1$.

Step 1: Stokes formula on Ω_r. — Since $(\Omega_r)^{\mathrm{cl}} \subset \Omega$, $u \in C^1((\Omega_r)^{\mathrm{cl}})$. Moreover, Ω_r is smooth. Hence, according to Stokes' formula,

$$\int_{\partial\Omega_r} v(z)\overline{u}(z)\,\mathrm{d}z = 2\mathrm{i}\int_{\Omega_r}\partial_{\overline{z}}(v\overline{u})(z)\,\mathrm{d}A(z) = 2\mathrm{i}\int_{\Omega_r} v(z)\overline{\partial_z u}(z)\,\mathrm{d}A(z), \tag{36}$$

where the last equality follows from the holomorphy of v.

Step 2: Limit of the right-hand side. — Since $\partial_z u \in L^2(\Omega)$ (by hypothesis) and $v \in C^0(\Omega^{\mathrm{cl}})$, this proves that $\mathbb{1}_{\Omega_r} v\overline{\partial_z u}$ is dominated by an $L^1(\Omega)$ function. Hence, the dominated convergence theorem gives

$$\int_{\Omega_r} v(z)\overline{\partial_z u}(z)\,\mathrm{d}A(z) = \int_\Omega \mathbb{1}_{\Omega_r} v(z)\overline{\partial_z u}(z)\,\mathrm{d}A(z) \xrightarrow[r\to 1]{} \int_\Omega v(z)\overline{\partial_z u}(z)\,\mathrm{d}A(z).$$

Step 3: Limit of the left-hand side. — For $|z| < 1$, let us set $U(z) = u(\varphi(z))$ and $V(z) = zv(\varphi(z))\varphi'(z)$.
We claim that

$$\int_{\partial\Omega_r}\overline{u}(z)v(z)\,\mathrm{d}z = \mathrm{i}\int_0^{2\pi}\overline{g}(\varphi(\mathrm{e}^{\mathrm{i}\theta}))v(\varphi(r^2\mathrm{e}^{\mathrm{i}\theta}))r^2\mathrm{e}^{\mathrm{i}\theta}\varphi'(r^2\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta. \tag{37}$$

Indeed, according to theorem B.1, the function $|\varphi'|$ is bounded from below by some constant $c > 0$, so that

$$\infty > \int_{\partial\Omega}|g(z)|^2|\mathrm{d}z| = \int_{\mathbb{T}}|g\circ\varphi(\zeta)|^2|\varphi'(\zeta)||\mathrm{d}\zeta| \geq c\int_{\mathbb{T}}|g\circ\varphi(\zeta)|^2|\mathrm{d}\zeta|.$$

Hence $g\circ\varphi \in L^2(\mathbb{T})$. Then, since U is harmonic in $\mathbb{D}$ with boundary values $g\circ\varphi$, and using $P^{\mathbb{D}}(z,\mathrm{e}^{\mathrm{i}\theta})$, the Poisson kernel on $\mathbb{D}$ already introduced earlier, we have

$$U(z) = \int_0^{2\pi} g(\varphi(\mathrm{e}^{\mathrm{i}\theta}))P(z,\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta.$$

On the other hand

$$\begin{aligned}\int_{\partial\Omega_r} \overline{u}(z)v(z)\,\mathrm{d}z &= \int_0^{2\pi} \overline{u}(\varphi(r\mathrm{e}^{\mathrm{i}t}))v(\varphi(r\mathrm{e}^{\mathrm{i}t}))r\mathrm{i}\mathrm{e}^{\mathrm{i}t}\varphi'(r\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}t \\ &= \mathrm{i}\int_0^{2\pi} \overline{U}(r\mathrm{e}^{\mathrm{i}t})V(r\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}t \\ &= \mathrm{i}\int_0^{2\pi}\int_0^{2\pi} \overline{g}(\varphi(\mathrm{e}^{\mathrm{i}\theta}))P(r\mathrm{e}^{\mathrm{i}t},\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta\, V(r\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}t.\end{aligned}$$

Since v is holomorphic, V is harmonic. Using Fubini's theorem, we recognise a Poisson integral in the t variable:

$$\begin{aligned}\int_{\partial\Omega_r} \overline{u}(z)v(z)\,\mathrm{d}z &= \mathrm{i}\int_0^{2\pi} \overline{g}(\varphi(\mathrm{e}^{\mathrm{i}\theta}))\int_0^{2\pi} \underbrace{P(r\mathrm{e}^{\mathrm{i}t},\mathrm{e}^{\mathrm{i}\theta})}_{=P(r\mathrm{e}^{\mathrm{i}\theta},\mathrm{e}^{\mathrm{i}t})} V(r\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}t\,\mathrm{d}\theta \\ &= \mathrm{i}\int_0^{2\pi} \overline{g}(\varphi(\mathrm{e}^{\mathrm{i}\theta}))V(r^2\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta,\end{aligned}$$

which is the claimed formula.

We now justify that the right-hand side of this formula converges thanks to the dominated convergence theorem. According to theorem B.1, there is a constant C such that for every $\mathrm{e}^{\mathrm{i}\theta} \neq \zeta_k$ ($k = 1, \dots, n$),

$$|\varphi'(r\mathrm{e}^{\mathrm{i}\theta})| \leq C|\varphi'(\mathrm{e}^{\mathrm{i}\theta})|.$$

The hypothesis $g \in L^2(\partial\Omega)$ reads $\int_0^{2\pi} |g(\varphi(\mathrm{e}^{\mathrm{i}\theta}))|^2|\varphi'(\mathrm{e}^{\mathrm{i}\theta})|\,\mathrm{d}\theta < +\infty$. Hence, the integrand of the right-hand side of eq. (37) is dominated by the L^1 function $C|g(\varphi(\mathrm{e}^{\mathrm{i}\theta}))||\varphi'(\mathrm{e}^{\mathrm{i}\theta})|$, and the dominated convergence theorem gives

$$\begin{aligned}\lim_{r\to 1}\int_{\partial\Omega_r} \overline{u}(z)v(z)\,\mathrm{d}z &= \mathrm{i}\int_0^{2\pi} \overline{g}(\varphi(\mathrm{e}^{\mathrm{i}\theta}))v(\varphi(\mathrm{e}^{\mathrm{i}\theta}))\mathrm{e}^{\mathrm{i}\theta}\varphi'(\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta \\ &= \int_{\partial\Omega} \overline{g}(z)v(z)\,\mathrm{d}z.\end{aligned}$$

Combining this limit with the limit on $\int_{\Omega_r} \partial_{\overline{z}}(\overline{u}v)$ of the previous step, the proposition is proved. □

C The half-heat equation on a segment

Let $\omega \subset (0,\pi)$ be non-trivial open segment. Here, we explain how to transfer some of our results on the half-heat equation on the torus (2) to the half-heat equation on the segment with Dirichlet boundary conditions:

$$\begin{cases} (\partial_t + |D_I|)f(t,x) = \mathbb{1}_\omega u(t,x), & t \geq 0,\ x \in (0,\pi), \\ f(t,0) = f(t,\pi) = 0, & t \geq 0, \\ f(0,x) = f_0(x), & x \in (0,\pi),\ f_0 \in L^2(0,\pi), \end{cases} \tag{38}$$

where $|D_I| = \sqrt{-\partial_x^2}$ with Dirichlet boundary conditions on $(0,\pi)$, defined by functional calculus, i.e.,

$$\begin{cases} D(|D_I|) = \left\{\sum_{n>0} f_n\sin(nx),\ \sum_{n>0}|n|^2|f_n|^2 < +\infty\right\} = W_0^{1,2}(0,\pi), \\ |D_I|\Big(\sum_{n>0} f_n\sin(nx)\Big) = \sum_{n>0} nf_n\sin(nx). \end{cases}$$

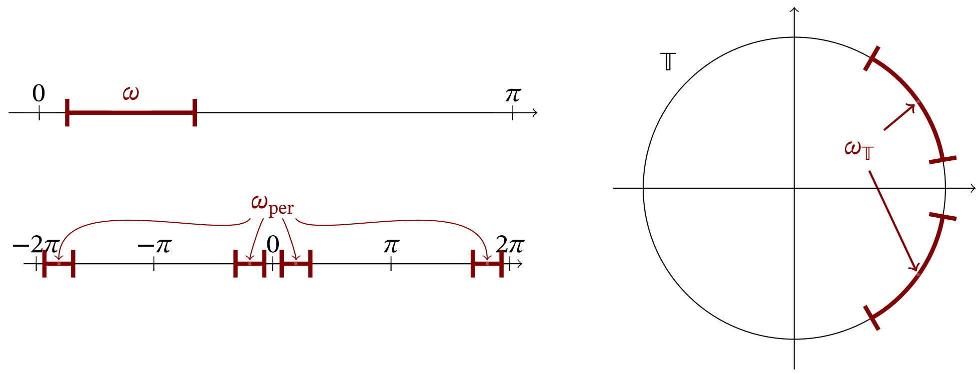

Figure 6: Top left: illustration of an example $\omega \subset (0, \pi)$. Right: the corresponding $\omega_{\mathbb{T}} \subset \mathbb{T}$ for the purpose of theorem C.2. Bottom left: the corresponding $\omega_{\text{per}} \subset \mathbb{R}$ for the purpose of theorem C.5. In general, $\omega_{\mathbb{T}}$ is not an interval, but the union of two intervals. While our results on $\mathcal{NC}_{H^2}(\omega')$ and $\mathcal{NC}_{L^2}(\omega')$ are mostly stated when ω' is an interval of $\mathbb{T}$, they also hold when ω' is a finite union of intervals, hence we will apply them with $\omega_{\mathbb{T}}$.

We will call this control system the Dirichlet control system.

Definition C.1. We say that $f_0 \in L^2(0, \pi)$ is null-controllable (or steerable to 0) from ω in time T if there exists $u \in L^2([0, T] \times \omega)$ such that the solution f of the Dirichlet control system (38) with initial condition f_0 is such that $f(T, \cdot) = 0$.

We denote by $\mathcal{NC}_I(\omega, T)$ the set of initial conditions that are steerable to 0 from ω in time T.

Theorem C.2. *For a function $f \in L^2(0, \pi)$, extended as an odd 2π-periodic function, let $\mathfrak{E} f(\mathrm{e}^{\mathrm{i}x}) = f(x)$. Define $\omega_{\mathbb{T}} := \{\zeta \in \mathbb{T}, |\arg(\zeta)| \in \omega\}$ (here,* arg *is the determination of the argument in $(-\pi, \pi]$, see fig. 6).*

Then, for any $f_0 \in L^2(0, \pi)$, $f_0 \in \mathcal{NC}_I(\omega, T)$ if and only if $\mathfrak{E} f_0 \in \mathcal{NC}_{L^2}(\omega_{\mathbb{T}})$.

Note that up to a $\sqrt{2}$ factor, $\mathfrak{E}$ is an isometry from $L^2(0, \pi)$ into $L^2(\mathbb{T})$. The proof of theorem C.2 relies on the following lemma, that shows this embedding $\mathfrak{E}$ is a natural tool to study the Dirichlet control system (38):

Lemma C.3. *Let $S(t)$ be the semigroup on $L^2(\mathbb{T})$ generated by $|D_x|$ (eq. (4)). Let $S_I(t)$ be the semigroup generated by $|D_I|$. Then $S(t) \circ \mathfrak{E} = \mathfrak{E} \circ S_I(t)$.*

Proof. If we write $f \in L^2(0, \pi)$ as $f(x) = \sum_{n>0} f_n \sin(nx)$ and $\mathfrak{E} f$ as $\mathfrak{E} f(\mathrm{e}^{\mathrm{i}x}) = \sum_{n \in \mathbb{Z}} \widehat{\mathfrak{E} f}(n) \mathrm{e}^{\mathrm{i}nx}$, it is standard that $f_n = 2\mathrm{i}\widehat{\mathfrak{E} f}(n) = -2\mathrm{i}\widehat{\mathfrak{E} f}(-n)$ and $\widehat{\mathfrak{E} f}(0) = 0$. The lemma is then a consequence of the formula for the semigroups:

$$\begin{aligned} S(t)\mathfrak{E} f(\mathrm{e}^{\mathrm{i}x}) &= \textstyle\sum_{n \in \mathbb{Z}} \widehat{\mathfrak{E} f}(n) \mathrm{e}^{-|n|t} \mathrm{e}^{\mathrm{i}nx}, \\ S_I(t) f(x) &= \textstyle\sum_{n>0} f_n \mathrm{e}^{-nt} \sin(nx). \end{aligned}$$

□

Proof of theorem C.2. As in the proof of theorem 1.2, $f_0 \in L^2(0, \pi)$ is steerable to 0 from ω in time T if and only if

$$\forall g_0 \in L^2(0, \pi),\ |\langle f_0, S_I(T) g_0 \rangle_{L^2(0,\pi)}| \lesssim \|S_I(t) g_0\|_{L^2([0,T] \times \omega)}.$$

Since $\mathfrak{E}$ is an isometry (up to some constant) from $L^2(0, \pi)$ into $L^2(\mathbb{T})$, but also from $L^2(\omega)$ into $L^2(\omega_{\mathbb{T}})$, this is equivalent to

$$\forall g_0 \in L^2(0, \pi),\ |\langle \mathfrak{E} f_0, \mathfrak{E} S_I(T) g_0 \rangle_{L^2(\mathbb{T})}| \lesssim \|\mathfrak{E} S_I(t) g_0\|_{L^2([0,T] \times \omega_{\mathbb{T}})}.$$

Using the lemma, this is in turn equivalent to

$$\forall g_1 \in \mathfrak{E}L^2(0,\pi),\ |\langle \mathfrak{E}f_0, S(T)g_1\rangle_{L^2(\mathbb{T})}| \lesssim \|S(t)g_1\|_{L^2([0,T]\times\omega_{\mathbb{T}})}.$$

Now, $\mathfrak{E}L^2(0,\pi)$ is the closed subspace of $L^2(\mathbb{T})$ consisting of the odd functions, and it is characterized by $g_1 \in \mathfrak{E}L^2(0,\pi)$ if and only if $\forall n \in \mathbb{Z}, \widehat{g_1}(-n) = -\widehat{g_1}(n)$. Hence, it is stable by $S(t)$. Moreover, every $g_0 \in L^2(\mathbb{T})$ can be written as $g_0 = g_1 + g_2$ with $g_1 \in \mathfrak{E}L^2(0,\pi)$ (i.e., g_1 odd) and g_2 even. The space of even functions is orthogonal to the space of odd functions, hence these functions g_1 and g_2 are orthogonal in $L^2(\mathbb{T})$. If $g_0 \in L^2(\omega)$, seen as function in $L^2(0,\pi)$ supported on ω, the corresponding functions g_1 and g_2 are orthogonal in $L^2(\omega_{\mathbb{T}})$. Thus, since $\mathfrak{E}f_0$ is odd by construction,

$$\begin{aligned}&\langle \mathfrak{E}f_0, S(T)g_1\rangle_{L^2(\mathbb{T})} = \langle \mathfrak{E}f_0, S(T)g_0\rangle_{L^2(\mathbb{T})},\\ &\|S(t)g_0\|^2_{L^2([0,T]\times\omega_{\mathbb{T}})} = \|S(t)g_1\|^2_{L^2([0,T]\times\omega_{\mathbb{T}})} + \|S(t)g_2\|^2_{L^2([0,T]\times\omega_{\mathbb{T}})}.\end{aligned}$$

Thus, f_0 is steerable to 0 from ω in time T if and only if

$$\forall g_0 \in L^2(\mathbb{T}),\ |\langle \mathfrak{E}f_0, S(T)g_0\rangle_{L^2(\mathbb{T})}| \lesssim \|S(t)g_0\|_{L^2([0,T]\times\omega_{\mathbb{T}})},$$

i.e., if and only if $\mathfrak{E}f_0$ is steerable to 0 from $\omega_{\mathbb{T}}$ in time T. □

To say something about the null-controllable states for the Dirichlet control system (38), one only has to use our theorems on the L^2 control system and apply them to $\mathfrak{E}f_0$. With one caveat: the majority of our theorems are stated when ω is an interval of $\mathbb{T}$, but in the context of theorem C.2, $\omega_{\mathbb{T}}$, might not be an interval, even if ω is an interval of $(0,\pi)$ (see fig. 6). Fortunately, our theorems do hold with the same proofs when ω is finite a union of open intervals of $\mathbb{T}$ that are at positive distance from one another. Hence we can in fact apply our theorems because $\omega_{\mathbb{T}}$ is the union of two intervals of $\mathbb{T}$ (and if they touch, it counts as one interval only). For instance:

Theorem C.4. *Let ω be a non-trivial open interval of $(0,\pi)$. For every $T, T' > 0$, $\mathcal{NC}_I(\omega,T) = \mathcal{NC}_I(\omega,T')$.*

Proof. This is an immediate consequence of our characterization of $\mathcal{NC}_I(\omega,T)$ (theorem C.2) and the independence of $\mathcal{NC}_{L^2}(\omega,T)$ with respect to time (theorem 1.12). □

Theorem C.5. *Let ω be a non-trivial open interval of $(0,\pi)$. Let $\omega_{\mathrm{per}} = \{x \in \mathbb{R},\ \mathrm{e}^{\mathrm{i}x} \in \omega_{\mathbb{T}}\}$ (i.e., the union of ω, $-\omega$ and their translations by $2\pi k$ for $k \in \mathbb{Z}$, see fig. 6). Let $f_0 \in L^2(0,\pi)$. Assume that $f_0 \in \mathcal{NC}_I(\omega)$. Then, there exists f_1 harmonic on $\mathbb{R}^2 \setminus \omega_{\mathrm{per}}^{\mathrm{cl}}$, 2π-periodic in x, bounded on $\{(x,y),\ |y|>1\}$ such that $f_1(x,0) = f_0(x)$ for almost every $x \in (0,\pi)\setminus\omega$.*

Note that this necessary condition implies that f_0 is analytic on $(0,\pi)\setminus\omega^{\mathrm{cl}}$, but also implies some compatibility conditions at the boundary (unless at the possible point where ω touches the boundary): for instance if 0 is not adherent to ω, since the odd extension of f_0 must be analytic at 0, this implies in particular that for $k \in \mathbb{N}$, $\partial_x^{2k} f(0) = 0$.

Let us recall the following fact that will be frequently used in the following proof: P_+g corresponds to the non-negative frequencies of g and $\overline{P_+\overline{g}}$ to the non-positive ones. In particular, $P_+g + \overline{P_+\overline{g}} = g + \widehat{g}(0)$.

Proof. Let $f_0 \in \mathcal{NC}_I(\omega)$. Thanks to our characterization of $\mathcal{NC}_I(\omega)$, we have $\mathfrak{E}f_0 \in \mathcal{NC}_{L^2}(\omega_{\mathbb{T}})$. Hence, according to our characterization of $\mathcal{NC}_{L^2}$ (theorem 1.11), $P_+\mathfrak{E}f_0 \in \mathcal{NC}_{H^2}(\omega_{\mathbb{T}})$ and $P_+\overline{\mathfrak{E}f_0} \in \mathcal{NC}_{H^2}(\omega_{\mathbb{T}})$. Finally, our necessary condition for $\mathcal{NC}_{H^2}$ (theorem 1.5) tells that there exists two functions $f_\pm$ holomorphic on $\widehat{\mathbb{C}}\setminus\omega_{\mathbb{T}}^{\mathrm{cl}}$ that coincide on $\mathbb{D}$ with $P_+\mathfrak{E}f_0$ and $P_+\overline{\mathfrak{E}f_0}$ respectively.

Set $f_1(x,y) := (f_+ + \overline{f_-})(\mathrm{e}^{\mathrm{i}x-y})$. By definition, f_1 is 2π-periodic in x. Note that $f_+ = P_+\mathfrak{C}f_0$ on $\mathbb{T}\setminus\omega_{\mathbb{T}}^{\mathrm{cl}}$ and $f_- = P_+\overline{\mathfrak{C}f_0}$ on $\mathbb{T}\setminus\omega_{\mathbb{T}}^{\mathrm{cl}}$. Note also that by definition, $\mathfrak{C}f_0$ is odd, hence $\widehat{\mathfrak{C}f_0}(0) = 0$. Hence, we get for $x\in\mathbb{R}\setminus\omega_{\mathrm{per}}^{\mathrm{cl}}$ (i.e., $\mathrm{e}^{\mathrm{i}x}\notin\omega_{\mathbb{T}}^{\mathrm{cl}}$)

$$\begin{aligned}
f_1(x,0) &= P_+\mathfrak{C}f_0(\mathrm{e}^{\mathrm{i}x}) + \overline{P_+\overline{\mathfrak{C}f_0}(\mathrm{e}^{\mathrm{i}x})}\\
&= \mathfrak{C}f_0(\mathrm{e}^{\mathrm{i}x}) + \widehat{\mathfrak{C}f_0}(0)\\
&= f_0(x).
\end{aligned}$$

Since $P_+\mathfrak{C}f_0$ and $P_+\overline{\mathfrak{C}f_0}$ are holomorphic on $\widehat{\mathbb{C}}\setminus\omega_{\mathbb{T}}^{\mathrm{cl}}$,

$$\begin{aligned}
\left(\frac{\mathrm{d}}{\mathrm{d}t} - \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}\right)\left(f_+(\mathrm{e}^{\mathrm{i}x-t})\right) &= (-\mathrm{e}^{\mathrm{i}x-t} - \mathrm{i}^2\mathrm{e}^{\mathrm{i}x-t})f_+'(\mathrm{e}^{\mathrm{i}x-t}) = 0,\\
\left(\frac{\mathrm{d}}{\mathrm{d}t} + \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}\right)\left(\overline{f_-}(\mathrm{e}^{\mathrm{i}x-t})\right) &= \overline{(-\mathrm{e}^{\mathrm{i}x-t} - \mathrm{i}^2\mathrm{e}^{\mathrm{i}x-t})f_-'(\mathrm{e}^{\mathrm{i}x-t})} = 0.
\end{aligned}$$

Hence,

$$\Delta f_1 = \left(\frac{\mathrm{d}}{\mathrm{d}t} + \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}\right)\left(\frac{\mathrm{d}}{\mathrm{d}t} - \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}\right)\left(f_+(\mathrm{e}^{\mathrm{i}x}) + \overline{f_-}(\mathrm{e}^{\mathrm{i}x-t})\right)\right) = 0.$$

That is, f_1 is harmonic on $\mathbb{R}^2\setminus\omega_{\mathrm{per}}^{\mathrm{cl}}$ and bounded away from the real axis. □

Corollary C.6. *Let ω be a non-trivial interval of $(0,\pi)$ that is at positive distance from $\{0,\pi\}$. Let f_0 be real-analytic on $(0,\pi)$. If $f_0\in\mathcal{NC}_I(\omega)$, then $f_0 = 0$.*

Proof. According to our characterization of $\mathcal{NC}_I(\omega)$ (theorem C.5), $\mathfrak{C}f_0\in\mathcal{NC}_{H^2}(\omega_{\mathbb{T}})$. Moreover, according to our hypothesis, $\mathfrak{C}f_0$ is real-analytic at every point of $\mathbb{T}\setminus\{1,-1\}$. And with theorem C.5, we can prove that $\mathfrak{C}f_0$ is also real-analytic at 1 and -1: indeed, let f_1 as in theorem C.5, i.e., for $x\in\mathbb{R}$, $\mathfrak{C}f_0(\mathrm{e}^{\mathrm{i}x}) = f_1(x,0)$. Since f_1 is harmonic (hence real-analytic) outside $\omega_{\mathrm{per}}^{\mathrm{cl}}\times\{0\}$, $\mathfrak{C}f_0$ is indeed real-analytic outside $\omega_{\mathbb{T}}^{\mathrm{cl}}h$ and in particular at 1 and -1 (because ω is at positive distance from $\{0,\pi\}$). In conclusion, $\mathfrak{C}f_0$ is real-analytic on $\mathbb{T}$. Hence, $\mathfrak{C}f_0 = 0$ (theorem 1.13). Hence $f_0 = 0$. □

Our proof does not work as-is if ω touches the boundary because we don't know whether $\mathfrak{C}f_0$ is analytic at the corresponding point. Maybe finer properties of harmonic functions could help, but we leave that for future work. In any case, if we assume a priori that f_0 is also analytic at 0, we recover the result.

Theorem C.7. *Let ω be a non trivial open interval of $(0,\pi)$ that stays at positive distance from $\{0,\pi\}$. Let $f_0\in W^{1/2,2}_{00}(0,\pi)$. Then $f_0\in\mathcal{NC}_I(\omega)$ if and only if there exists $g\in W^{1/2,2}_{00}\big(\omega\cup(-\omega)\big)$ such that $\int_{-\pi}^{\pi} g(x)\,\mathrm{d}x = 0$ and*

$$f_0(x) = \sum_{n>0}\hat{g}(n)\sin(nx),\quad \textit{with}\ \ \hat{g}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)\mathrm{e}^{-\mathrm{i}nx}\,\mathrm{d}x.$$

Proof. Since $f_0\in W^{1/2,2}_{00}(0,\pi)$, by definition of $W^{1/2,2}_{00}(0,\pi)$ the extension by 0 is $W^{1/2,2}(\mathbb{R})$. Substracting $f_0(-x)$, we see that $\mathfrak{C}f_0\in W^{1/2,2}(\mathbb{T})$.

Combining our characterization of $\mathcal{NC}_I(\omega)$ (theorem C.2), of $\mathcal{NC}_{L^2}(\omega)$ (theorem 1.11) and of $\mathcal{NC}_{H^2}(\omega)\cap W^{1/2,2}(\mathbb{T})$ (theorem 1.9), we see that $f_0\in\mathcal{NC}_I(\omega)$ if and only if there exists $g_\pm\in W^{1/2,2}_{00}(\omega_{\mathbb{T}})$ such that

$$P_+\mathfrak{C}f_0 = P_+g_+,\quad P_+\overline{\mathfrak{C}f_0} = P_+g_-. \tag{39}$$

Step 1: Necessary condition. — We claim that $g_-(\mathrm{e}^{\mathrm{i}x}) = -\overline{g_+}(\mathrm{e}^{-\mathrm{i}x})$.

Since $\mathfrak{C}f_0$ is odd, $\widehat{\mathfrak{C}f_0}(0) = 0$ and

$$\mathfrak{C}f_0 = P_+\mathfrak{C}f_0 + \overline{P_+\overline{\mathfrak{C}f_0}} = P_+g_+ + \overline{P_+g_-}.$$

Using again that $\mathfrak{C}f_0$ is odd (hence $\widehat{\mathfrak{C}f_0}(-n) = -\widehat{\mathfrak{C}f_0}(n)$), we deduce that for $n > 0$,

$$-\widehat{g_+}(n) = -\widehat{\mathfrak{C}f_0}(n) = \widehat{\mathfrak{C}f_0}(-n) = \overline{\widehat{g_-}(n)}.$$

Hence, if we set $r(\mathrm{e}^{\mathrm{i}x}) = \overline{g_-(\mathrm{e}^{\mathrm{i}x})} + g_+(\mathrm{e}^{-\mathrm{i}x})$, we have

$$r(\mathrm{e}^{\mathrm{i}x}) = \sum_{n\in\mathbb{Z}} \overline{\widehat{g_-}(n)}\mathrm{e}^{-\mathrm{i}nx} + \widehat{g_+}(n)\mathrm{e}^{-\mathrm{i}nx}.$$

According to the previous computation, all the terms for $n \geq 0$ vanishes. Hence, r only has positive frequencies, and since $r \in L^2(\mathbb{T})$, we get $r \in H^2$. Since $\omega_{\mathbb{T}}$ is symetric (see fig. 6), $g_+(\mathrm{e}^{-\mathrm{i}x})$ is also supported on $\omega_{\mathbb{T}}$. Hence, $\operatorname{supp}(r) \subset \omega_{\mathbb{T}}^{\mathrm{cl}}$. Thanks to the unique continuation property of H^2 functions (see [25, §I.1.1.2] or [45, Theorem 17.18]), $r = 0$ and the claim is proved.

In particular, the relation $-\widehat{g_+}(n) = \overline{\widehat{g_-}(n)}$ stays true for every $n \in \mathbb{Z}$. Thus, we get

$$\begin{aligned}
f_0(x) &= \mathfrak{C}f_0(\mathrm{e}^{\mathrm{i}x}) \\
&= P_+g_+(\mathrm{e}^{\mathrm{i}x}) + \overline{P_+g_-}(\mathrm{e}^{\mathrm{i}x}) \\
&= \sum_{n\geq 0} \widehat{g_+}(n)\mathrm{e}^{\mathrm{i}nx} + \overline{\widehat{g_-}(n)}\mathrm{e}^{-\mathrm{i}nx} \\
&= \sum_{n\geq 0} \widehat{g_+}(n)(\mathrm{e}^{\mathrm{i}nx} - \mathrm{e}^{-\mathrm{i}nx}) \\
&= \sum_{n>0} 2\mathrm{i}\widehat{g_+}(n)\sin(nx).
\end{aligned}$$

Thus, the necessary condition holds with $g(x) = 2\mathrm{i}g_+(\mathrm{e}^{\mathrm{i}x})$.

Step 2: Sufficient condition. — Let g as in the statement. Then

$$f_0(x) = \frac{1}{2\mathrm{i}}\sum_{n>0} \widehat{g}(n)(\mathrm{e}^{\mathrm{i}nx} - \mathrm{e}^{-\mathrm{i}nx}) = \frac{1}{2\mathrm{i}}\sum_{n>0} \widehat{g}(n)(\mathrm{e}^{\mathrm{i}nx} - \mathrm{e}^{-\mathrm{i}nx}).$$

Thus, with $g_+(\mathrm{e}^{\mathrm{i}x}) = g(x)/2\mathrm{i}$, we have $P_+\mathfrak{C}f_0 = P_+g_+$, and with $g_-(\mathrm{e}^{\mathrm{i}x}) = g_+(\mathrm{e}^{-\mathrm{i}x})$, we have $P_+\overline{\mathfrak{C}f_0} = -P_+g_-$. Hence $\mathfrak{C}f_0 \in \mathcal{NC}_{L^2}(\omega)$. ☐

D The reachable space

Some of our methods can be used to study the space of *reachable states*, defined as follows:

Definition D.1. We say that $f_T \in H^2$ (resp. $L^2(\mathbb{T})$) is *reachable on ω in time T on H^2* (resp. L^2) if there exists a control $u \in L^2([0,T]\times\omega)$ such that the solution f of the control system (3) (resp. (2)) with initial condition 0 satisfies $f(T,\cdot) = f_T$.

We denote the set of reachable functions on ω in time T by $\mathcal{R}_{H^2}(\omega, T)$ (resp. $\mathcal{R}_{L^2}(\omega, T)$).

With the methods of section 3, we could prove results on $\mathcal{R}_{L^2}(\omega, T)$ from results on $\mathcal{R}_{H^2}(\omega, T)$. We do not detail this further, and instead sketch some results on $\mathcal{R}_{H^2}(\omega, T)$.

Proposition D.2. *Let ω be an open subset of $\mathbb{T}$ and $T > 0$. Let $\Omega_T^* := \{z \in \mathbb{C},\ \mathrm{e}^{-T} < |z| < 1,\ z/|z| \in \omega\}$ (see fig. 1). Let $f_T \in H^2$. The following assertions are equivalent:*

- *$f_T \in \mathcal{R}_{H^2}(\omega, T)$.*
- *For every polynomial $g \in \mathbb{C}[X]$,*

$$|\langle f_T, g\rangle_{H^2}| \lesssim \|g\|_{A^2(\Omega_T^*)}. \tag{40}$$

The proof is very similar to the one of eq. (5), with the appropriate observability inequality.

Theorem D.3. *Let $T > 0$ and ω be an open subset of $\mathbb{T}$. Let Ω_T as in theorem 1.2. If $f_T \in \mathcal{R}_{H^2}(\omega)$, f_T can be extended holomorphically on $\widehat{\mathbb{C}} \setminus (\Omega_T)^{\mathrm{cl}}$.*

The proof is very similar to the one of theorem 1.4 and is omitted.

We proved that every function in $P_+(W_{00}^{1/2,2}(\omega))$ is null-controllable to 0. In fact, the same strategy based on Stokes' formula used in section 2.4.1 proves these functions are also reachable:

Theorem D.4. *Let $T > 0$ and ω be an interval of $\mathbb{T}$. If $h \in W_{00}^{1/2,2}(\omega)$, we implicitly extend h on $\mathbb{T}$ by 0 outside ω. Then,*

$$P_+(W_{00}^{1/2,2}(\omega)) \subset \mathcal{R}_{H^2}(\omega, T).$$

We might be able to adapt the methods used in the proof of theorem 1.7 to get a space of reachable states with less regularity, but we leave that for another time. However, we can exhibit explicit functions in $\mathcal{R}_{H^2}(\omega, T)$ that are not null-controllable:

Theorem D.5. *Let $T > 0$ and ω be an open subset of $\mathbb{T}$. Let Ω_T^* as in theorem D.2. For $|u| < 1$, let $k_u(z) = 1/(1 - \bar{u}z)$. Then,*

$$k_u \in \mathcal{R}_{H^2}(\omega, T) \Leftrightarrow u \in \Omega_T^*.$$

In addition, $k_u \notin \mathcal{NC}_{H^2}(\omega)$.

Proof. Since k_u is the reproducing kernel of H^2, the left-hand side of the observability inequality (40) is

$$\langle k_u, g\rangle_{H^2} = g(u).$$

This can be estimated by $\|g\|_{A^2(\Omega_T^*)}$ if and only if $u \in \Omega_T^*$.

Since k_u is holomorphic on $D(0, 1/|u|)$, it cannot be steered to 0 (theorem 1.6). □

In particular $\mathcal{R}_{H^2}(\omega, T)$ genuinely depends on T. This is coherent with the following observation:

Proposition D.6.

$$\mathcal{NC}_{H^2}(\omega) = S(T)^{-1}(\mathcal{R}_{H^2}(\omega, T)).$$

Proof. Let us denote by $S(t, f_0, u)$ the solution of the half-heat equation (3). According to Duhamel's formula, $S(t, f_0, u) = S(t)f_0 + \mathcal{F}_t^+ u$. Then, by definition of $\mathcal{NC}_{H^2}(\omega)$ and $\mathcal{R}_{H^2}(\omega, T)$,

$$\begin{aligned} f_0 \in \mathcal{NC}_{H^2}(\omega) &\Leftrightarrow \exists u \in L^2([0, T] \times \omega),\ S(T, f_0, u) = 0 \\ &\Leftrightarrow \exists u \in L^2([0, T] \times \omega),\ S(T)f_0 = -\mathcal{F}_T^+ u \\ &\Leftrightarrow S(T)f_0 \in \mathcal{R}_{H^2}(\omega, T). \end{aligned}$$

□

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