(Non) Null Controllability of Some Degenerate Parabolic Equations

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Dispersive waves and related topics
Conference in honor of Gilles Lebeau

The Problem of Null-Controllability

Definition of the Null-Controllability

Ω domain of $\mathbb{R}^n$, ω an open subset of Ω and $T > 0$.

Definition (Controllability of the heat equation on ω in time T)
For every initial condition $f_0 \in L^2(\Omega)$, there exists a control $u \in L^2([0, T] \times \omega)$ such that the solution f of:

$$\partial_t f - \Delta f = 1_\omega u, \quad f_{|\partial\Omega} = 0, \quad f(0) = f_0$$

satisfies $f(T) = 0$.

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Theorem (Controllability of the heat equation (Lebeau & Robbiano 1995, Fursikov & Imanuvilov 1996))
Ω a C^2 bounded domain of $\mathbb{R}^n$, ω a non empty open subset of Ω, and $T > 0$. The heat equation is null-controllable on ω in time T.

Some element of proofs

Observability: dual notion to null-controllability

Theorem

- *The equation* $\partial_t - \Delta f = 1_\omega u$ *is null controllable* is equivalent to
- *For every solution of* $\partial_t g - \Delta g = 0$,

$$|g(T,\cdot)|^2_{L^2(\Omega)} \leq C|g|^2_{L^2([0,T]\times\omega)}.$$

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Theorem (Abstract linear control system)

- *The equation* $\partial_t f + Af = Bu$ *is null controllable* is equivalent to
- *For every solution of* $\partial_t g + A^* g = 0$,

$$|g(T)|^2_H \leq C \int_0^T |B^* g(t)|^2_U \,\mathrm{d}t.$$

Spectral Inequality

Theorem (Spectral inequality, Lebeau & Robbiano 1995)
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ϕ_k eigenfunction of $-\Delta$, eigenvalue λ_k.

$$\Big|\sum_{\lambda_k \leq \mu} a_k \phi_k\Big|_{L^2(\Omega)} \leq C e^{K\sqrt{\mu}} \Big|\sum_{\lambda_k \leq \mu} a_k \phi_k\Big|_{L^2(\omega)}$$

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- Allows us to bring components for $\lambda_k \leq \mu$ to 0.
- Dissipation of the heat equation : $f_0 = \sum_{\lambda_k > \mu} a_k \phi_k$

$$|e^{t\Delta} f_0|^2_{L^2(\Omega)} \leq e^{-2\mu t}|f_0|^2_{L^2(\Omega)}$$

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- Dissipation $\gg$ spectral inequality $\implies$ controllability
- Only depends on the spectral inequality
- E.g. also works for $\partial_t + (-\Delta)^\alpha$ (with $\alpha > 1/2$)

Examples of parabolic PDEs with few dissipation

Examples

- Fractional heat $(\partial_t + (-\Delta)^\alpha)f = 1_\omega u \quad (\alpha \le 1/2)$
  Spectral inequality with $\sqrt{\mu}$, Dissipation with μ^α
- Grushin $(\partial_t - \partial_x^2 - x^2\partial_y^2)f = 1_\omega u$
  Spectral inequality with μ, Dissipation with μ
- Kolmogorov $(\partial_t - \partial_v^2 - v^2\partial_x)f = 1_\omega u$
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Null-Controllable ?

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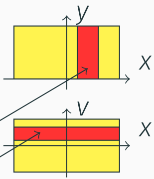
- Grushin [BCG BMM BDE]: only in large time if ω
- Kolmogorov [BZ B BHHR]: only in large time ω



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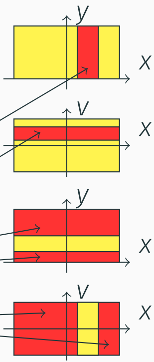
- Grushin: never null-controllable if ω
- Kolmogorov: never null-controllable if ω



Half Heat Equation

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Observability inequality applied with $g(t,y) = \sum_{n>0} a_n e^{-nt} e^{iny}$:

$$\sum_{n>0} |a_n|^2 e^{-2nT} \leq C \int_{[0,T]\times\omega} \left| \sum_{n>0} a_n e^{-nt} e^{iny} \right|^2 \mathrm{d}t \, \mathrm{d}y$$

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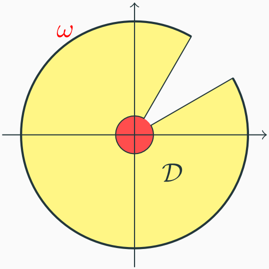
Let $\zeta = e^{-t+iy}$ and $f(\zeta) = \sum a_n \zeta^{n-1}$

$$\begin{aligned} &\int_{D(0,e^{-T})} |f(\zeta)|^2 \,\mathrm{d}\lambda(\zeta) \\ &\leq 2\pi \sum_{n>0} |a_n|^2 e^{-2nT} \\ &\leq 2\pi C \int_{\mathcal{D}} |f(\zeta)|^2 \,\mathrm{d}\lambda(\zeta) \end{aligned}$$



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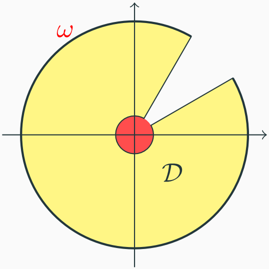

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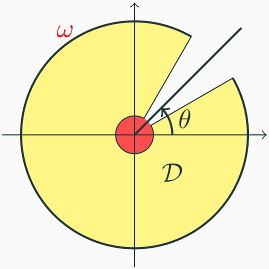

Untrue thanks to Runge's theorem (take $f_k \longrightarrow 1/\zeta$ uniformly on every compact of $\mathbb{C} \setminus e^{i\theta}\mathbb{R}_+$) □

Explicit non-null controllable data
ω strict open subset of $\mathbb{T}$.

$c > 0$ and $f_0(x) = \sum_{n\in\mathbb{Z}} a_n e^{inx}$ with $|a_n| < c^{-1}e^{-c|n|}$.

f_0 is null-controllable if and only if $f_0 = 0$.

Half-heat equation : more results (without proofs)

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Rotated half-heat equation
$\Omega = \mathbb{T}$ or $\mathbb{R}$, $\Re(z) > 0$, ω strict open subset of Ω.

The rotated half-heat equation $(\partial_t + z\sqrt{-\Delta})f(t,x) = 1_\omega u$ is never null-controllable on ω.

Grushin equation

Equation
$(\partial_t - \partial_x^2 - x^2\partial_y^2)f(t,x,y) = 1_\omega u$ on $(-1,1)\times\mathbb{T}$ (Dirichlet BC)

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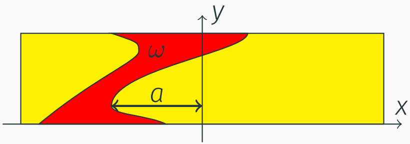
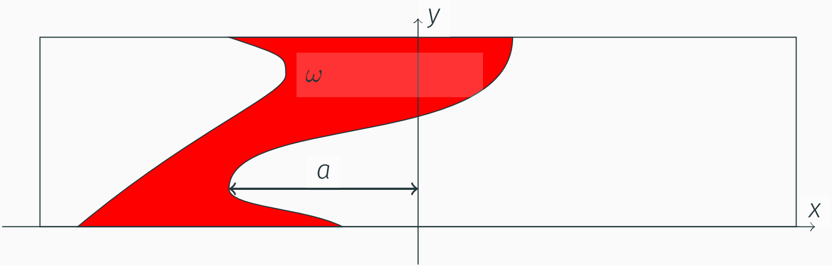
Theorem (Duprez-K)

$\omega = \{\gamma_1(y) < x < \gamma_2(y)\}$
$a = \max(\sup(\gamma_2^-), \sup(\gamma_1^+))$



- *The Grushin equation is null-controllable if $T > a^2/2$.*
- *The Grushin equation is not null-controllable if $T < a^2/2$.*

Grushin: null-controllability in large time



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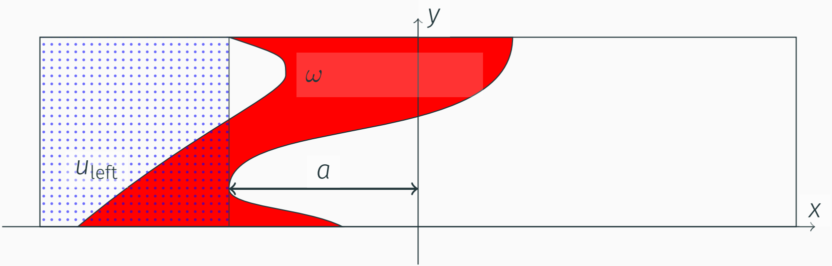

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- u_{left} controls f_0 from a band on the left (possible if $T > a^2/2$)

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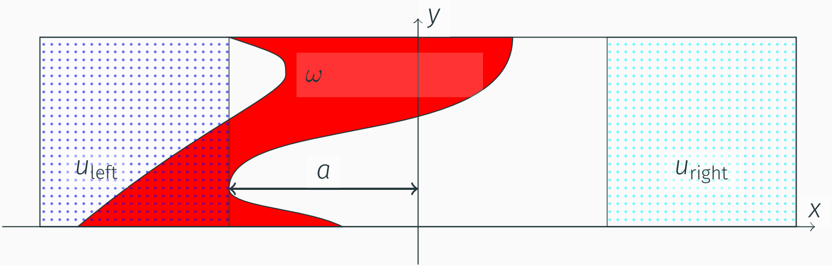

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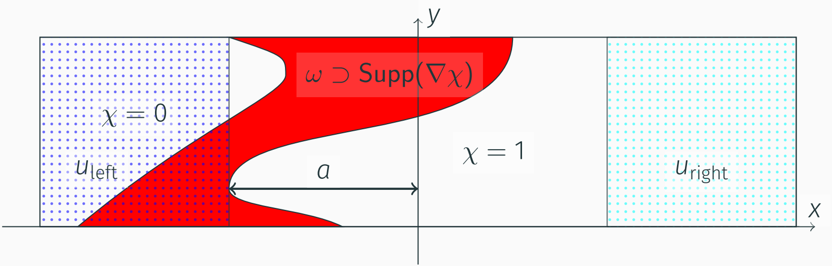

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- u_{left} controls f_0 from a band on the left (possible if $T > a^2/2$)
- u_{right} controls f_0 from a band on the right (possible if $T > a^2/2$)
- χ cutoff with $\mathrm{Supp}(\nabla\xi) \subset \omega$, $\chi = 0$ “left of ω” and $\chi = 1$ “right of ω”
- $f := \chi f_{\mathrm{left}} + (1-\chi) f_{\mathrm{right}}$.
  $(\partial_t - \partial_x^2 - x^2\partial_y^2) f = \chi u_{\mathrm{left}} + (1-\chi) u_{\mathrm{right}} +$ some things with $\nabla\chi, \Delta\chi$

Grushin: Lack of null controllability in small time

Observability inequality for the Grushin equation

$(\partial_t - \partial_x^2 - x^2\partial_y^2)g(t,x,y) = 0 \implies |g(T,\cdot,\cdot)|_{L^2(\mathbb{R}\times\mathbb{T})} \leq C|g|_{L^2([0,T]\times\mathbb{R}\times\omega_y)}$

Grushin and half-heat equation

$g(t,x,y) = \sum_{n>0} a_n e^{-nt} e^{-nx^2/2+iny}$. In y variable: looks like half-heat.

Subcritical fractional heat equation and Kolmogorov equation

Fractional Heat Equation ($\alpha < 1/2$, $\Re(z) > 0$)

Non-null-controllability of $\partial_t + z(-\Delta)^\alpha$ on $\Omega = \mathbb{R}$, $\omega = \{|x| > \epsilon\}$

- Controllability $\Leftrightarrow$ observability:
  $(\partial_t + \bar{z}(-\Delta)^\alpha)g = 0 \implies |g(T,\cdot)|_{L^2(\Omega)} \leq C|g|_{L^2([0,T]\times\omega)}$

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$$g_h(t,x) = c_h \int_{\mathbb{R}} \chi(\xi - \xi_0)e^{-(\xi-\xi_0)^2/2h + ix\xi/h - t\bar{z}|\xi|^{2\alpha}/h^{2\alpha}}\mathrm{d}\xi$$

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- Saddle point method:

$$g_h(t,x) = \mathcal{O}\left(\frac{1}{|x|^\infty}e^{-c/h-cth^{-2\alpha}}\right) \qquad |x| > \epsilon$$

$$g_h(t,x) = e^{ix\xi_0/h-x^2/2h-\mathcal{O}(h^{-2\alpha})} \qquad |x| < \frac{\xi_0}{4}$$

Kolmogorov equation

Observability inequality for the Kolmogorov equation

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Kolmogorov and fractional heat equation

$$g(t,x,v) = \sum_{n>0} a_n e^{-\sqrt{in}\,t} e^{-\sqrt{in}\,v^2/2+inx}$$

In x variable, looks like $(\partial_t + \sqrt{i}(-\partial_x^2)^{1/4})g(t,x) = 0$.

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Counter-example to the observability inequality
$\tilde{g}_h(t,x)$ previous counter-example to

$$(\partial_t + \sqrt{i}(-\partial_x^2)^{1/4})\tilde{g}(t,x) = 0 \implies |\tilde{g}(T,\cdot)|_{L^2(\mathbb{T})} \leq C|\tilde{g}|_{L^2([0,T]\times\omega_x)}$$

choose $g_h(t,x,v) = \tilde{g}_h(t+v^2/2,x)$.

Conclusion

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That's all folks!