Control of a degenerate parabolic equation: minimal time and geometric dependance

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The Problem of Null-Controllability

Definition of the Null-Controllability

Ω domain of $\mathbb{R}^n$, ω an open subset of Ω and $T > 0$.

Definition (Controllability of the heat equation on ω in time T)
For every initial condition $f_0 \in L^2(\Omega)$, there exists a control $u \in L^2([0, T] \times \omega)$ such that the solution f of:

$$\partial_t f - \Delta f = 1_\omega u, \quad f_{|\partial\Omega} = 0, \quad f(0) = f_0$$

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Controllability of the heat equation [Lebeau & Robbiano 1995, Fursikov & Imanuvilov 1996]
Ω a C^2 bounded domain of $\mathbb{R}^n$, ω a non empty open subset of Ω, and $T > 0$. The heat equation is null-controllable on ω in time T.

Observability: dual notion to null-controllability

Theorem

- *The equation* $\partial_t - \Delta f = 1_{\omega} u$ *is null controllable* is equivalent to
- *For every solution of* $\partial_t g - \Delta g = 0$,

$$|g(T,\cdot)|^2_{L^2(\Omega)} \leq C|g|^2_{L^2([0,T]\times\omega)}.$$

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Theorem (Abstract linear control system)

- *The equation* $\partial_t f + Af = Bu$ *is null controllable* is equivalent to
- *For every solution of* $\partial_t g + A^* g = 0$,

$$|g(T)|^2_H \leq C \int_0^T |B^* g(t)|^2_U \,\mathrm{d}t.$$

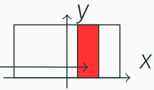
A degenerate parabolic equation: the Grushin equation

Grushin equation

$$(\partial_t - \partial_x^2 - x^2\partial_y^2)f(t,x,y) = 1_\omega u(t,x,y) \quad \text{Dirichlet BC on } (-1,1)\times\mathbb{T}$$

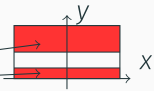
Controllable ?

- Only in large time if ω



  [Beauchard-Cannarsa-Guglielmi 2014, Beauchard-Miller-Morancey 2015, Beauchard-Dardé-Erverdoz 2018]

- [K 2017] Never null-controllable if ω



Examples

Grushin equation

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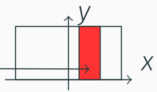

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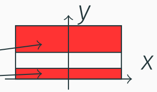
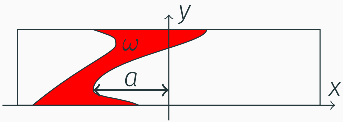


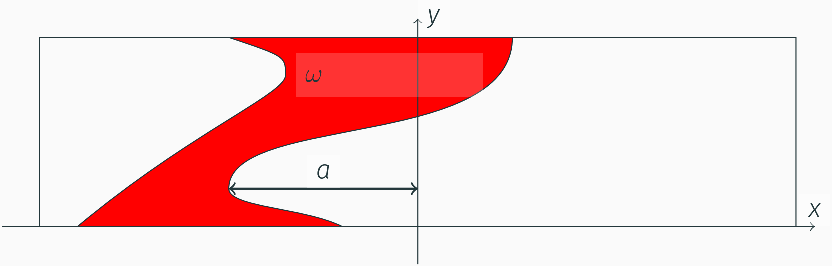
Theorem [Duprez-K 2018]

$\omega = \{\gamma_1(y) < x < \gamma_2(y)\}$
$a = \max(\sup(\gamma_2^-), \sup(\gamma_1^+))$



- The Grushin equation is null-controllable if $T > a^2/2$.
- The Grushin equation is not null-controllable if $T < a^2/2$.

Grushin: null-controllability in large time



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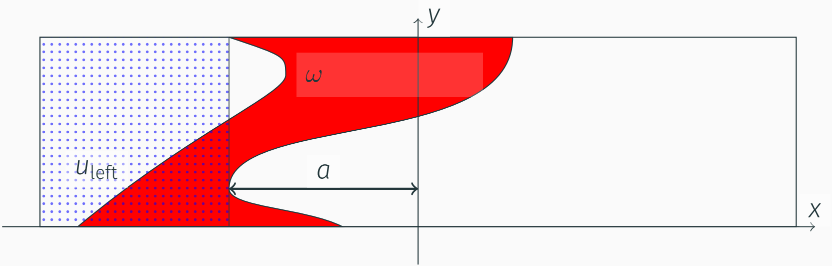

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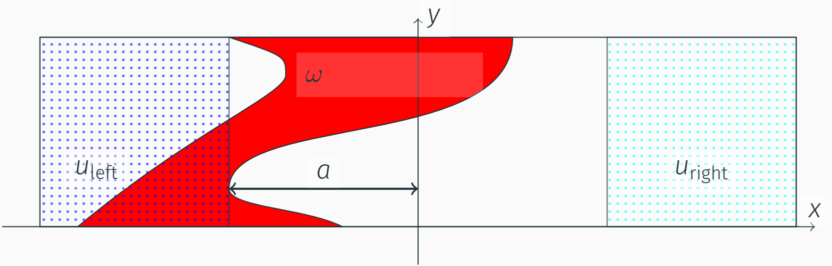

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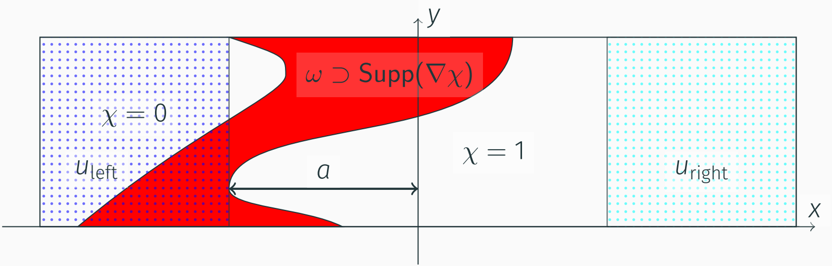

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- u_{left} controls f_0 from a band on the left (possible if $T > a^2/2$)
- u_{right} controls f_0 from a band on the right (possible if $T > a^2/2$)
- χ cutoff with $\text{Supp}(\nabla\chi) \subset \omega$, $\chi = 0$ “left of ω” and $\chi = 1$ “right of ω”
- $f := \chi f_{\text{left}} + (1-\chi) f_{\text{right}}$.
  $(\partial_t - \partial_x^2 - x^2\partial_y^2) f = \chi u_{\text{left}} + (1-\chi) u_{\text{right}} +$ some things with $\nabla\chi, \Delta\chi$

Grushin: Lack of null controllability in small time

Observability inequality for the Grushin equation
$(\partial_t - \partial_x^2 - x^2\partial_y^2)g(t,x,y) = 0 \implies |g(T,\cdot,\cdot)|^2_{L^2(\mathbb{R}\times\mathbb{T})} \leq C|g|^2_{L^2([0,T]\times\omega)}$

Particular solutions
$g(t,x,y) = \sum_{n>0} a_n e^{-nt} e^{-nx^2/2+iny}$

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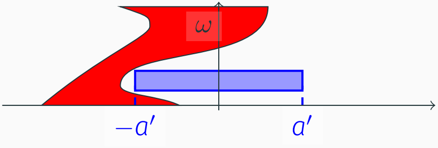

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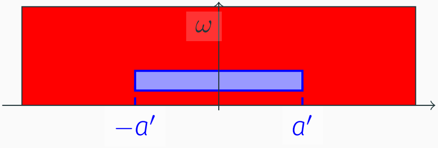

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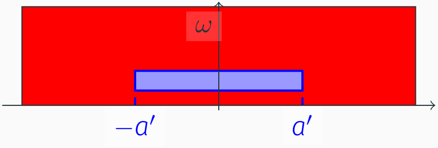

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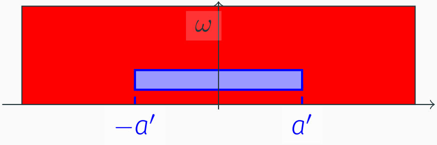
Upper bound RHS
Change of variables $z_x = e^{-t+iy-x^2/2}$
$p(z) = \sum a_n z^{n-1}$
$\text{RHS} = \int_{\mathbb{R}} \int_{\mathcal{D}_x} |p(z_x)|^2 \,\mathrm{d}\lambda(z_x)\,\mathrm{d}x$

$\leq \sup_{z\in\bigcup_{x\in\mathbb{R}}\mathcal{D}_x} C|p(z)|^2$

$\mathcal{D}_x = \{z_x, 0 < t < T, (x,y) \in \omega\}$



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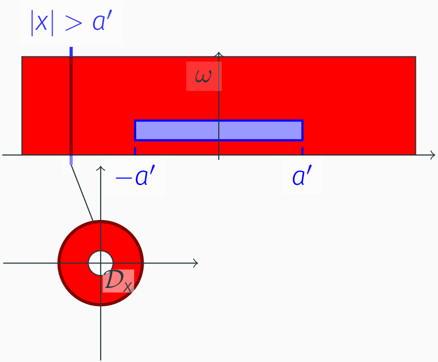

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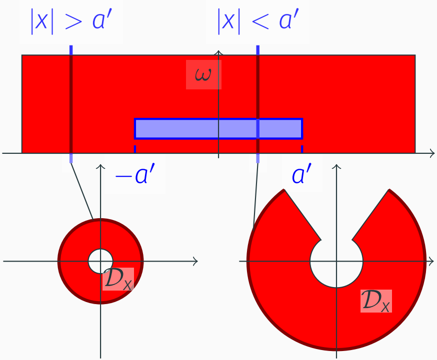

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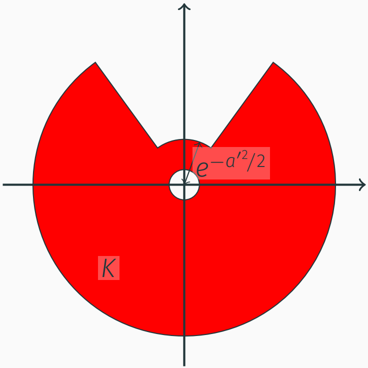
$\text{RHS} \leq \sup_K C|p(z)|^2$



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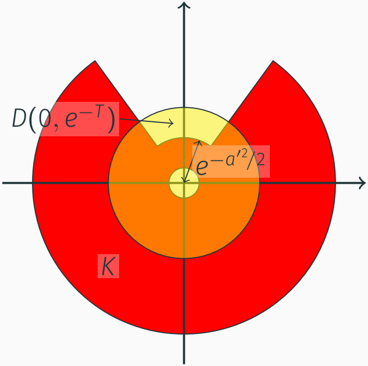

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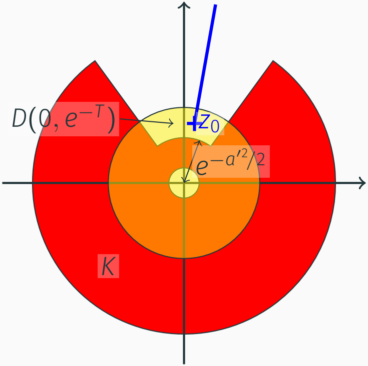
$$\text{LHS} \geq \int_{D(0,e^{-T})} |p(z)|^2 \, \mathrm{d}z$$

Polynomial estimate

$|p|_{L^2(D(0,e^{-T}))} \leq C|p|_{L^\infty(K)}$

Does not hold

(take $p_k(z) \to (z - z_0)^{-1}$)



Conclusion

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That's all folks!