Quadratic Obstruction for the Local Controllability of a Water-Tank System and the KdV Equation

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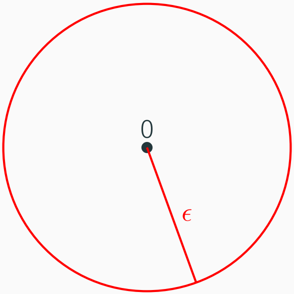
Workshop TRECOS 2022

Introduction

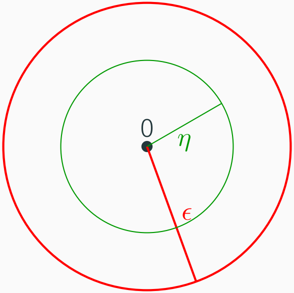
Small-time Local Controllability

$\dot{X} = f(X, u)$ with $f(0, 0) = 0$.



Small-time Local Controllability

$\dot{X} = f(X, u)$ with $f(0, 0) = 0$. For $\epsilon > 0$



Small-time Local Controllability

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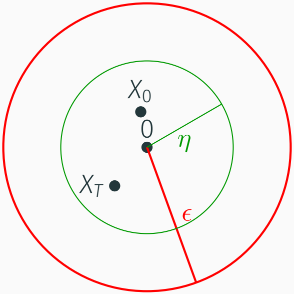
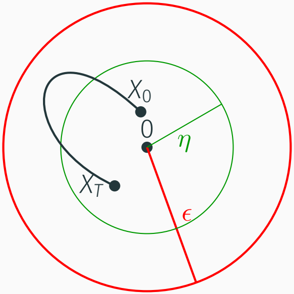


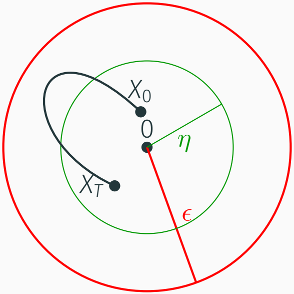
Small-time Local Controllability

$\dot{X} = f(X,u)$ with $f(0,0) = 0$. For $\epsilon > 0$, does there exists $\eta > 0$ such that if $|T| < \epsilon$, $|X_0| < \eta$, $|X_T| < \eta$



Small-time Local Controllability

$\dot{X} = f(X, u)$ with $f(0, 0) = 0$. For $\epsilon > 0$, does there exists $\eta > 0$ such that if $|T| < \epsilon$, $|X_0| < \eta$, $|X_T| < \eta$, we can find $|u|_{L^\infty(0,T)} < \epsilon$ such that $X(T) = X_T$?



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Theorem

Small-time local controllability does hold if the linearized equation is null-controllable.

The converse is not true.

A simple quadratic obstruction

$$\begin{cases} \dot{x}_1 = u \\ \dot{x}_2 = x_1^2 \end{cases}$$

$\dot{x}_2 \geq 0$: no controllability.

A quadratic obstruction in small time

$$\begin{cases} \dot{x}_1 = u \\ \dot{x}_2 = x_1 \\ \dot{x}_3 = x_1^2 - x_2^2 \end{cases}$$

If $x_2(0) = x_2(T) = 0$, $\int_0^T x_2^2 \leq (T/\pi)^2 \int_0^T \dot{x}_2^2$ (Poincaré). If T is small, $x_3(T) \geq x_3(0)$: no small-time controllabillity

Another small-time obstruction?

$$\begin{cases} \dot{x}_1 = u \\ \dot{x}_2 = x_1 \\ \dot{x}_3 = x_1^3 + x_2^2 \end{cases}$$

Small-time local controllability... but not if we ask $|u|_{W^{1,\infty}} \ll 1$!

[Beauchard-Marbach, Quadratic obstructions to small-time local controllability for scalar-input systems, 2018,...]

Outline

Quadratic Obstruction for some PDEs

Burgers Equation

$$\partial_t f - \partial_{xx} f + f\partial_x f = u(t), \quad (t,x) \in (0,T) \times (0,1)$$

Nonlinear equation not small-time locally controllable. [Marbach 2018]

Schrödinger equation with bilinear controls

$$i\partial_t f = -\partial_x^2 f - u(t)\mu(x)f, \quad (t,x) \in (0,T) \times (0,1)$$

For some μ, local controllability around the ground state in large enough time, but no small-time local controllability. [Beauchard-Morancey 2014, Bournissou 2021 ... (see talk this afternoon)]

Nonlinear heat equation with bilinear controls

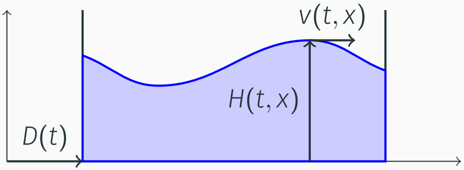
$$\partial_t f = -\partial_x^2 f - u(t)\Gamma[f], \quad (t,x) \in (0,T) \times (0,1)$$

For some nonlinearities Γ, no small-time local controllability (and/or other weird behaviour). [Beauchard-Marbach 2018]

Control of a Water-Tank

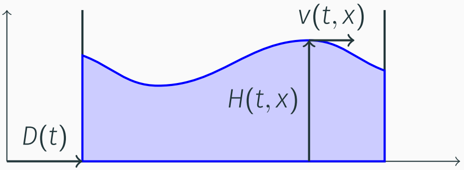
The water-tank system

$$\begin{cases} \partial_t H + \partial_x (vH) = 0, & (t,x) \in (0,T) \times (0,L) \\ \partial_t v + \partial_x (gH + v^2/2) = -u(t), & (t,x) \in (0,T) \times (0,L) \\ v(t,0) = v(t,L) = 0 & t \in (0,T) \\ \ddot{D}(t) = u(t) & t \in (0,T) \end{cases}$$



The water-tank system

$$\begin{cases} \partial_t H + \partial_x(vH) = 0, & (t,x) \in (0,T) \times (0,L) \\ \partial_t v + \partial_x(gH + v^2/2) = -u(t), & (t,x) \in (0,T) \times (0,L) \\ v(t,0) = v(t,L) = 0 & t \in (0,T) \\ \ddot{D}(t) = u(t) & t \in (0,T) \end{cases}$$



Linearized equation around $H = H_{\mathrm{eq}}$, $v = 0$

$$\begin{cases} \partial_t h + H_{\mathrm{eq}} \partial_x v = 0, & (t,x) \in (0,T) \times (0,L) \\ \partial_t v + g\partial_x h = -u(t), & (t,x) \in (0,T) \times (0,L) \\ v(t,0) = v(t,L) = 0 & t \in (0,T) \end{cases}$$

$h(t, L-x) = -h(t,x)$, $v(t, L-x) = v(t,x)$; not controllable. But moving the tank and such the water is still at the start and end is possible if $T > T_* = L/\sqrt{gH_{\mathrm{eq}}}$.

Theorem (Control using the return method, Coron 2002)

Local controllability il large time: there eists $T > 0$, $\eta > 0$ such that if

$$
\begin{aligned}
&|H_0 - 1|_{C^1} + |v_0|_{C^1} < \eta, \\
&|H_1 - 1|_{C^1} + |v_1|_{C^1} < \eta, \\
&|D_1 - D_0| < \eta
\end{aligned}
$$

then there exists a trajectory such that $H(t=0) = H_0$, $H(t=T) = H_1$, $v(t=0) = v_0$, $v(t=T) = v_1$, $D(0) = D_0$, $D(T) = D_1$, $\dot{D}(0) = \dot{D}(T) = 0$.

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Theorem (Lack of local controllability when the time is not large enough, Coron-K-Nguyen 2021)

For $T < 2T_*$, lack of local controllaility with controls small in C^0: there exists $\eta > 0$ such that if $H(t=0) = H(t=T) = H_{\text{eq}}$, $v(t=0) = v(t=T) = 0$, $\dot{D}(0) = \dot{D}(T) = 0$, and if $|u|_{C^0} < \eta$, then $u = 0$.

Proof strategy: $(H, v) \approx$ linearized + quadratic, and the quadratic term is $\geq c|u|^2_{H^{-1}}$.

Rescalling

$L = 1$, $H_{\text{eq}} = 1$, $g = 1$, $T_* = 1$.

Linéarised equation

$$\partial_t h_1 + \partial_x v_1 = 0$$
$$\partial_t v_1 + \partial_x h_1 = -u(t)$$
$$v_1(t, 0) = v_1(t, 1) = 0$$

Rescalling
$L = 1$, $H_{\mathrm{eq}} = 1$, $g = 1$, $T_* = 1$.

Quadratic term

$$\begin{aligned} \partial_t h_2 + \partial_x v_2 &= -\partial_x(h_1 v_1) \\ \partial_t v_2 + \partial_x h_2 &= -\partial_x(v_1^2/2) \\ v_2(t,0) = v_2(t,1) &= 0 \end{aligned}$$

Lemma

$$(h_2(T,\cdot),\phi) + (v_2(T,\cdot),\psi) = \int_{[0,T]^2} K_{T,\phi,\psi}(s_1,s_2)u(s_1)u(s_2)\,\mathrm{d}s_1\,\mathrm{d}s_2$$

for some explicitly computable kernel $K_{T,\phi,\psi}$.

Formula for the kernel (do not read)
With $\Phi(x) = (\phi(x) + \psi(x))/2$ for $0 < x < 1$ and $(\phi(-x) - \psi(-x))/2$ for $-1 < x < 0$,
$2K_{T,\phi,\psi}(s_1, s_2) =$

$$\begin{cases} \int_{-2T+2s_2}^{0} \Phi(s + T - s_2)\,\mathrm{d}s + 2(T - s_2)\Phi(T - s_2) - 4(T - s_2)\Phi(T - s_1) \\ \qquad \text{if } 2T - 1 < s_1 + s_2 < 2T \\ \int_{s_2 - s_1}^{2-2T+s_2+s_1} \Phi(s - s_2 + T)\,\mathrm{d}s + (4T - 1 - 3s_2 - s_1)\Phi(T - s_2) - (1 + 2T - 3s_2 + s_1)\Phi(T - s_1) \\ \qquad \text{if } 2T - 2 < s_1 + s_2 < 2T - 1 \\ \int_{21-2T+2s_2}^{0} \Phi(s + T - s_2)\,\mathrm{d}s + (1 + 2T - 2s_2)\Phi(T - s_2) - (-1 + 4T - 4s_2)\Phi(T - s_1) \\ \qquad \text{if } 2T - 3 < s_1 + s_2 < 2T - 2 \\ \int_{s_2 - s_1}^{4-2T+s_2+s_1} \Phi(s + T - s_2)\,\mathrm{d}s + (-2 + 4T - 3s_2 - s_1)\phi(T - s_2) - (2 + 2T - 3s_2 + s_1)\phi(T - s \\ \qquad \text{si } 2T - 4 < s_1 + s_2 < 2T - 3 \end{cases}$$

Lemma

$\Phi(x) = (\phi(x) + \psi(x))/2$ for $0 < x < 1$ and $(\phi(-x) - \psi(-x))/2$ for $-1 < x < 0$. If $1 < T < 2$ and if the control u steers the linearized equation from 0 to 0 (apart from maybe moving the tank),

$$(h_2(T,\cdot),\phi) + (v_2(T,\cdot),\psi) = \int_{[0,T-1]^2} K^{\mathrm{red}}_{T,\phi,\psi}(s_1,s_2)u(s_1)u(s_2)\,\mathrm{d}s_1\,\mathrm{d}s_2$$

with

$$K^{\mathrm{red}}_{T,\phi,\psi}(s_1,s_2) = \frac{3}{2}(1-|s_2-s_1|)\left(\overline{\Phi}(T-s_1\vee s_2) - \overline{\Phi}(T-s_1\wedge s_2)\right)$$

Choice of Φ:
Φ 1-periodic, $\Phi(s) = s$ for $s \in [1, T]$.

$$K^{\text{red}}_{T,\phi,\psi}(s_1, s_2) = \frac{3}{2}(-|s_2 - s_1| + (s_2 - s_1)^2)$$

Lemma

If $\int_0^{T-1} u(s)\,\mathrm{d}s = 0$, *and* $U(s) = \int_0^s u(s')\,\mathrm{d}s'$,

$$\int_{[0,T-1]^2} K^{\text{red}}_{T,\phi,\psi}(s_1, s_2) u(s_1) u(s_2)\,\mathrm{d}s_1\,\mathrm{d}s_2 = 3\int_0^{T-1} (U(s))^2\,\mathrm{d}s - 3\left(\int_0^{T-1} U(s)\,\mathrm{d}s\right)^2.$$

Proof.
Integrate by parts in s_1 and s_2. $\partial_{s_1 s_2} K^{\text{red}}_{T,\phi,\psi} = 3\delta_{s_1=s_2} - 3.$ □

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Integrate by parts in s_1 and s_2. $\partial_{s_1 s_2} K^{\mathrm{red}}_{T,\phi,\psi} = 3\delta_{s_1 = s_2} - 3$. □

Proposition

For $1 < T < 2$ *and* $U(s) = \int_0^s u(s')\,\mathrm{d}s'$

$$(h_2(T,\cdot), \phi) + (v_2(T,\cdot), \psi) \geq 3(2 - T)|U|^2_{L^2(0,T-1)}$$

The situation so far

- $(h, v) \approx \underbrace{(h_1, v_1)}_{\text{linear in } u} + \underbrace{(h_2, v_2)}_{\text{quadratic in } u}$
- If $(h_1, v_1)(T, \cdot) = 0$ and $1 < T < 2$, some scalar product $(h_2, v_2)(T, \cdot)$ is $\geq c|U|^2_{L^2}$.

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Proof of lack of local controllability.

- If u steers the nonlinear equation from 0 to 0, find $\tilde{u}$ close to u that steers the *linearized* equation from 0 to 0: $|U - \tilde{U}|_{L^2} \leq C|U|_{L^2}|u|_{C^0}$.
- $|(h, v)(u) - (h_1, v_1)(u) - (h_2, v_2)(u)|_{H^{-2}} \leq C|U|_{L^2}^2|u|_{C^0}$
- If $|u|_{C^0}$ is small enough, the error between $(h, v)(u)$ and $(h_2, v_2)(\tilde{u})$ cannot counter the positivity of $(h_2(\tilde{u}, t, \cdot), \phi) + (v_2(\tilde{u}, t, \cdot), \psi)$. $\square$

Control of the KdV Equation

KdV equation

$$\begin{cases} \partial_t y + \partial_x y + \partial_x^3 y + y\partial_x y = 0, & (t,x) \in (0,T)\times(0,L) \\ y(t,0) = y(t,L) = 0, \partial_x y(t,L) = u(t) & t \in (0,T) \end{cases}$$

KdV equation linearized around 0

$$\begin{cases} \partial_t y_1 + \partial_x y_1 + \partial_x^3 y_1 = 0, & (t,x) \in (0,T)\times(0,L) \\ y_1(t,0) = y_1(t,L) = 0, \partial_x y_1(t,L) = u(t) & t \in (0,T) \end{cases}$$

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Theorem (Rosier 1997)

The linearized KdV equation is controllable in some time (equivalently in arbitrarily small time) iff $L \notin \mathcal{N} := \left\{ 2\pi\sqrt{\frac{k^2+kl+l^2}{3}}, (k,l) \in (N^*)^2 \right\}$.

If $L \in \mathcal{N}$*, there is some finite dimensional unreachable space* $\mathcal{M}$.

Theorem (Rosier 1997)

If $L \notin \mathcal{N}$, the nonlinear KdV equation is small-time local controllable.

Theorem (Coron and Crépeau 2004)

If L can be written in a unique way as $L = 2\pi\sqrt{\frac{k^2+kl+l^2}{3}}$ and that $k = l$, the nonlinear KdV equation is small-time local controllable.

Theorem (Cerpa 2007, Crépeau and Cerpa 2009)

If $L \in \mathcal{N}$, there exists $T > 0$ such that the nonlinear KdV equation is locally controllable in time T.

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If L can be written in a unique way as $L = 2\pi\sqrt{\frac{k^2+kl+l^2}{3}}$ and that $k = l$, the nonlinear KdV equation is small-time local controllable.

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If $L \in \mathcal{N}$, there exists $T > 0$ such that the nonlinear KdV equation is locally controllable in time T.

Theorem (Coron K Nguyen 2020)

If $k \neq l \in \mathbb{N}^*$, $L = 2\pi\sqrt{\frac{k^2+kl+l^2}{3}}$ and $2k + l \notin 3\mathbb{N}$, lack of small-time local controllable of the nonlinear KdV equation for H^3 initial conditions with controls small in $H^1(0, T)$.

Order 2

$$\begin{cases} \partial_t y_1 + \partial_x y_1 + \partial_x^3 y_1 = 0, & (t,x) \in (0,T) \times (0,L) \\ y_1(t,0) = y_1(t,L) = 0,\ \partial_x y_1(t,L) = u(t) & t \in (0,T) \end{cases}$$

Lemma

If $\dim(\mathcal{M}) = 2$*, we identify* $\mathcal{M} \approx \mathbb{C}$*, and then for some explicit* $p \in \mathbb{R}$ *and function* ϕ*.*

$$y_{2|\mathcal{M}}(t) = \int_0^L \int_0^t y_1(s,x)^2 e^{ip(t-s)} \phi(x) \,\mathrm{d}x \,\mathrm{d}s.$$

Order 2

$$\begin{cases} \partial_t y_2 + \partial_x y_2 + \partial_x^3 y_2 = -y_1 \partial_x y_1, & (t,x) \in (0,T) \times (0,L) \\ y_2(t,0) = y_2(t,L) = \partial_x y_2(t,L) = 0 & t \in (0,T) \end{cases}$$

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Theorem

If $L = 2\pi\sqrt{\frac{k^2+kl+l^2}{3}}$ *with* $2k + l \notin 3\mathbb{N}$, *if* T *is small and if* u *steers* y_1 *from* 0 *to* 0*,*

$$y_{2|\mathcal{M}} = \int_0^L \int_0^T y_1(s,x)^2 e^{ip(T-s)} \phi(x)\,\mathrm{d}x\,\mathrm{d}s = EN(u)^2(1 + O(T^{1/4}))$$

where $E \in \mathbb{C}$ *and* $N(u) \sim \|u\|_{H^{-2/3}}$.

Proof.

- Take Fourier transform in t. For some explicitly computable function $\Lambda(x,z)$,

$$\hat{y}(z,x) = \hat{u}(z)\Lambda(z,x)$$

- Paley-Wiener: if, u steers the linearized equation from 0 to 0 then $\hat{u}$ and $\Lambda(\cdot,x)\hat{u}(\cdot)$ are entire and $|\hat{u}(z)| + |\hat{u}(z)\partial_x\Lambda(z,0)| \leq Ce^{T|\Im(z)|}$.
- Computations $y_{2|\mathcal{M}} = \int \hat{u}(s)\overline{\hat{u}(s-p)}B(s)\,\mathrm{d}s, \quad B(s) \underset{s\to\pm\infty}{\sim} E|s|^{-4/3}$
- In the integral above, the part for $|s| \leq m$ is $\leq CmT^{1/2}\|u\|^2_{H^{-2/3}}$ (we use the Paley-Wiener property here). □

End of the proof of the lack of local controllability

- The coercivity property tells us that the second order “drifts” in the non-reachable space $\mathcal{M}$.
- Choose y_0 along that direction, assume you can steer it to 0
- This control is close to another control that steers the linearized equation from 0 to 0
- Estimating the difference between the non linear solution and the second-order approximation
- Quadratic drift bigger than the error (if control small in regular enough norm) □

Conclusion

Water-tank

- A trajectory which is natural for the water-tank is possible for the linearized equation but not for the nonlinear equation.
- Minimal time for the local-controllability to hold?

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KdV

- For some critical lengths, lack of small-time local controllability for controls small in H^1.
- Small-time local controllability with less regular controls?
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That’s all folks!

Bonus: Coercivity of an arbitrary scalar product for the water tank

Question

Coercivity of Q_Ψ:

$$Q_\Psi(u) = \int_{[a,b]^2} u(s_1)u(s_2)(1+\epsilon|s_2-s_1|)\big(\Psi(s_1\wedge s_2)-\Psi(s_1\vee s_2)\big)\,\mathrm{d}s_1\,\mathrm{d}s_2?$$

(with $\Psi = -\Phi(T-s)$, $Q_\Psi = \langle\Phi$, order 2 for the water-tank$\rangle$.)

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(with $\Psi = -\Phi(T-s)$, $Q_\Psi = \langle\Phi, \text{order 2 for the water-tank}\rangle$.)

Lemma

$\Psi \in C^1$, $\Psi' \geq c > 0$. *Then,*

$$Q_\Psi(U') \geq \alpha |U|^2_{L^2} \quad \textit{for every } U \in H^1_0(a,b)$$

iff

$$\int_a^b \Psi'(s)\,\mathrm{d}s \int_a^b \frac{1}{\Psi'(s)}\,\mathrm{d}s < (b-a+\epsilon^{-1})^2$$

Proof.
Integrate by parts; consider the resulting formula as a quadratic form on $L^2(\Psi'(s)\,\mathrm{d}s)$; see that on a stable space with codimension 2, $Q_\Psi = \text{Identity}$; compute explicitly the 2×2 matrix on the orthogonal and study its positivity. □