Geometric control conditions for the fractional heat equation

Joint work with Paul Alphonse

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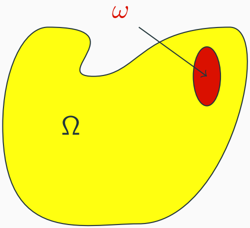
IMT, Université Toulouse III - Paul Sabatier

Introduction

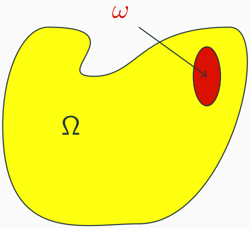


Definition (Null-controllability of the heat equation on ω in time T)

For every initial condition $f_0 \in L^2(\Omega)$, there exists a control $u \in L^2([0,T] \times \omega)$ such that the solution f of:

$$\partial_t f - \Delta f = \mathbf{1}_\omega u, \quad f_{|\partial\Omega} = 0, \quad f(0) = f_0$$

satisfies $f(T, \cdot) = 0$ on Ω.



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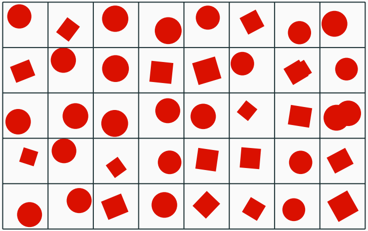
Theorem (Null-controllability of the heat equation (Lebeau & Robbiano 1995, Fursikov & Imanuvilov 1996))

Ω a C^2 connected bounded open subset of $\mathbb{R}^n$, ω a non-empty open subset of Ω, and $T > 0$. The heat equation is null-controllable on ω in time T.



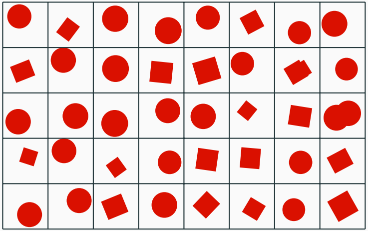
Definition ((γ, r)-thick set)

ω is (γ, r)-thick if for all x,

$$|B(x,r) \cap \omega| \geq \gamma |B(x,r)|$$

Theorem (Null-controllability of the heat equation on $\mathbb{R}^n$ (Egidi & Veselic 2018, Wang, Wang, Zhang & Zhang 2019))

$\omega \subset \mathbb{R}^n$ thick, and $T > 0$. The heat equation on $\mathbb{R}^n$ is null-controllable on ω in time T.



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Theorem (Kovrijkine's inequality)

If ω is (γ, r)-thick and $\operatorname{Supp} \hat{f} \subset B(0, \lambda)$, $\|f\|_{L^2(\mathbb{R}^n)} \leq \left(\frac{K_n}{\gamma}\right)^{K_n(1+r\lambda)} \|f\|_{L^2(\omega)}$

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Proposition (Duality null-controllability/observability)

Null-controllability in time T $\iff \forall g_0,\ \|e^{T\Delta} g_0\|^2_{L^2(\mathbb{R}^n)} \leq C \|e^{t\Delta} g_0\|^2_{L^2([0,T]\times\omega)}$

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Lebeau-Robbiano's method

- Kovrijkine's inequality $\implies$ null-controllability of $(\partial_t - \Delta)\Pi_\lambda f = \Pi_\lambda \mathbf{1}_\omega u$ (with $\Pi_\lambda =$ projection on frequencies $\leq \lambda$) with $\|u\|^2_{L^2} \leq \frac{C}{T} e^{C\lambda} \|f_0\|^2_{L^2}$

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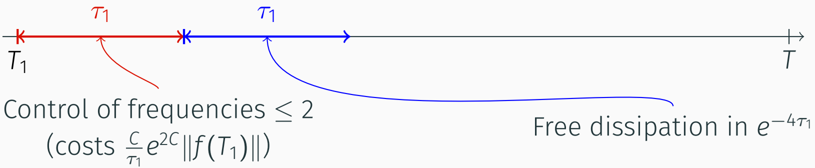
- 

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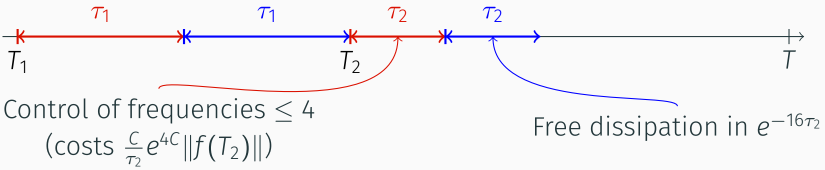
- 

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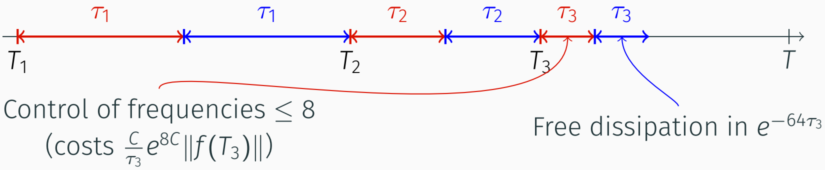
- 

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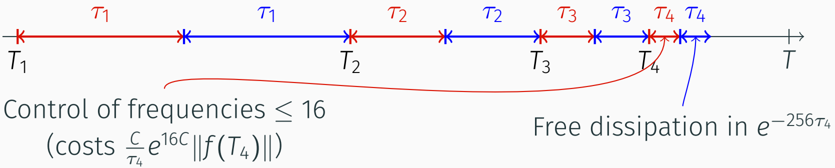
- 

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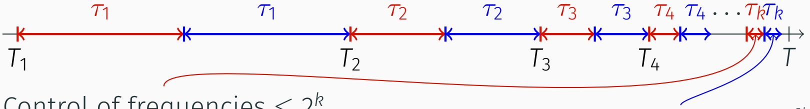
- 

Control of frequencies $\leq 2^k$ (costs $\frac{C}{\tau_k} e^{C 2^k} \|f(T_k)\|$) weaker than Free dissipation in $e^{-2^{2k}\tau_k}$

Fractional heat equation

$$(\partial_t + |D_x|^s)f(t,x) = \mathbf{1}_\omega u(t,x)$$

$$\widehat{|D_x|^s f}(\xi) = |\xi|^s \widehat{f}(\xi)$$

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Theorem (Case $s > 1$ (Alphonse & Bernier 2020, Alphonse & Martin 2023))

The fractional heat equation is null-controllable on ω in time $T \iff \omega$ is thick

Theorem (Case $s < 1$ (K 2020))

The fractional heat equation is null-controllable on ω in time $T \implies \omega$ is dense

Question

How dense must be ω to ensure the null-controllabilty of the fractional heat equation?

Null-controllability of the weakly dissipative fractional heat equation

Definition (Thick set with respect to a function)

$\gamma : (0, r_0] \to [0,1]$. ω is thick with respect to γ if for all x and $0 < r \leq r_0$.

$$|B(x,r) \cap \omega| \geq \gamma(r)|B(x,r)|$$

Theorem (Sufficient condition (Alphonse-K 2023))

Assume that ω is thick with respect to $\gamma_\alpha(r) := ce^{-Cr^{-\alpha}}$ for some $c, C > 0$ and $\alpha < s$.
The fractional heat equation is null-controllable on ω in time T

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Idea of the proof

- Kovrijkine: if $\operatorname{Supp} \hat{f} \subset B(0,\lambda)$ and $0 < r \leq r_0$,
  $\|f\|_{L^2(\mathbb{R}^n)} \leq \left(\frac{K_n}{\gamma_\alpha(r)}\right)^{K_n(1+r\lambda)} \|f\|_{L^2(\omega)}$
- Optimizing in r: if $\operatorname{Supp} \hat{f} \subset B(0,\lambda)$, $\|f\|_{L^2(\mathbb{R}^n)} \leq Ce^{C\lambda^\alpha} \|f\|_{L^2(\omega)}$
- Dissipation in $e^{-t\lambda^s}$: null-controllability thanks to Lebeau-Robbiano's method

Theorem (Necessary condition (Alphonse-K 2023))

Assume that the fractional heat equation is null controllable on ω in time T. Then, for some $C, c > 0$, ω is dense with respect to $ce^{-Cr^{-2s/(1-s)}}$.

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Coherent states

$$g_h(t,x) := e^{-t|D_x|^s} e^{ix\xi_0/h - x^2/2h} = c_{h,n} \int_{\mathbb{R}^n} e^{-(h\xi-\xi_0)^2/2h + ix\xi - t|\xi|^s} \, \mathrm{d}\xi$$

Asymptotics for coherent states

Saddle point method: $g_h(t,x) \approx c'_{h,n} e^{ix\xi_0/h - x^2/2h - t(\xi_0+ix)^s/h^s}$.

Proof

- Null-controllability $\iff$ observability $\implies$ $\|g_h(T,\cdot)\|^2_{L^2(\mathbb{R}^n)} \leq C\|g_h\|^2_{L^2([0,T]\times\omega)}$
- $e^{-2T|\xi_0|^s/h^s} \leq C(\underbrace{e^{-r^2/h}}_{\text{Contribution of } |x|>r} + \underbrace{|\omega \cap B(0,r)|}_{\text{Contribution of } |x|<r})$ (plus several error terms)
- Choose $h = h(r)$ small enough to absorb the $e^{-r^2/2h}$ into the left-hand-side

Examples of thick set with respect to a function

Definition (Smith-Volterra-Cantor set)

Algorithm:

- Start with $K_0 = [0, 1]$.
- K_n: remove a fraction τ_n of each interval that makes K_{n-1}

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Fraction removed: $\tau_1 = 0.5$

Smith-Volterra-Cantor sets

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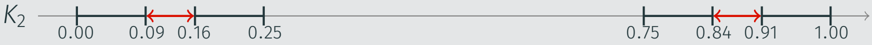
Fraction removed: $\tau_2 = 0.25$



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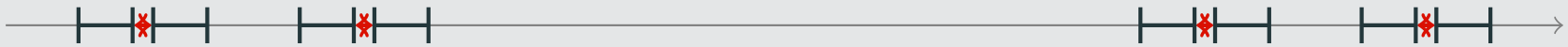
Fraction removed: $\tau_3 = 0.125$

K_3



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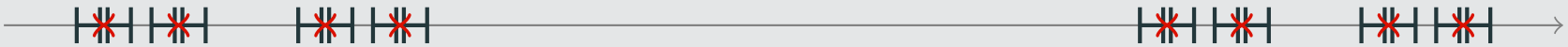
Fraction removed: $\tau_4 = 0.0625$

K_4 

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Construction of thick set with respect to a function γ: $\omega = \mathbb{R} \setminus K$

$$\forall r \text{ small enough, } \overbrace{\frac{1}{24} \sum_{n \geq \log_2(3|K|/r)} \tau_n}^{\geq \gamma(r) \text{ for well chosen } \tau_n} \leq \inf_{x \in \mathbb{R}} \frac{|\omega \cap B(x, r)|}{|B(x, r)|}$$

That’s all folks!

More general Fourier multipliers

Heat-like equation

$$\partial_t f(t,x) + \rho(|D_x|)f(t,x) = \mathbf{1}_\omega u(t,x) \qquad (E_\rho)$$

with $\rho\colon \mathbb{R}^+ \to \mathbb{C}$ measurable, $\operatorname{Re}\rho \geq 0$

Theorem

$0 < \operatorname{Re}\rho(\xi) \xrightarrow[\xi\to+\infty]{} +\infty$. $\gamma_\rho(r) := c_0 \exp(-c_1 \operatorname{Re}\rho(1/r)^\alpha)$, *for some* $c_0 \in (0,1)$, $c_1 > 0$ *and* $\alpha \in (0,1)$. *Let* ω *be thick relatively to* γ_ρ. *For every* $T > 0$, *the parabolic equation* (E_ρ) *is null-controllable on* ω *in time* T.

More general Fourier multipliers

Heat-like equation

$$\partial_t f(t,x) + \rho(|D_x|)f(t,x) = 1_\omega u(t,x) \tag{E_ρ}$$

with $\rho\colon \mathbb{R}^+ \to \mathbb{C}$ measurable, $\operatorname{Re}\rho \geq 0$

Theorem

Let $K > 0$ and $\mathcal{C} = \{\xi \in \mathbb{C}, \operatorname{Re}(\xi) > K, |\operatorname{Im}(\xi)| < K^{-1}\operatorname{Re}(\xi)\}$. Assume

$$\rho \text{ holomorphic on } \mathcal{C} \quad \rho(\xi) \underset{|\xi|\to+\infty}{=} o(\xi) \quad |\operatorname{Im}\rho(\xi)| \leq C\operatorname{Re}\rho(\xi) \quad \ln(\xi) \underset{|\xi|\to+\infty}{=} o(\operatorname{Re}\rho(\xi))$$

If (E_ρ) is null-controllable on ω in time $T > 0$, $\exists\lambda > 0$

$$\frac{|\omega \cap B(x,r)|}{|B(x,r)|} \geq Cr^{-n}\exp\left(-2(T+\epsilon)\operatorname{Re}\rho\left(\frac{\lambda}{h_r}\right)\right)$$

where h_r is chosen such that $\sqrt{h_r(2T+\epsilon)\operatorname{Re}\rho\left(\frac{\lambda}{h_r}\right)} \leq r$