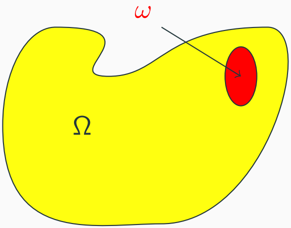
Null-controllability of parabolic-transport systems

Armand Koenig

20th June 2023

Workshop ANR TRECOS

Introduction

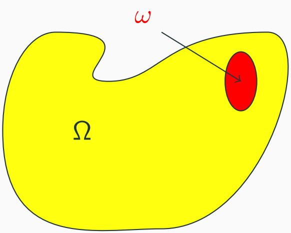


Definition (Null-controllability of the heat equation on ω in time T)

For every initial condition $f_0 \in L^2(\Omega)$, there exists a control $u \in L^2([0,T] \times \omega)$ such that the solution f of:

$$\partial_t f - \Delta f = 1_\omega u, \quad f_{|\partial\Omega} = 0, \quad f(0) = f_0$$

satisfies $f(T, \cdot) = 0$ on Ω.



Definition (Null-controllability of the heat equation on ω in time T)

For every initial condition $f_0 \in L^2(\Omega)$, there exists a control $u \in L^2([0,T] \times \omega)$ such that the solution f of:

$$\partial_t f - \Delta f = \mathbb{1}_\omega u, \quad f_{|\partial\Omega} = 0, \quad f(0) = f_0$$

satisfies $f(T, \cdot) = 0$ on Ω.

Theorem (Null-controllability of the heat equation (Lebeau & Robbiano 1995, Fursikov & Imanuvilov 1996))

Ω a C^2 connected bounded open subset of $\mathbb{R}^n$, ω a non-empty open subset of Ω, and $T > 0$. The heat equation is null-controllable on ω in time T.

The equation:

$$\partial_t f(t,x) + A\partial_x f(t,x) - B\partial_x^2 f(t,x) + Kf(t,x) = M\mathbf{1}_\omega u(t,x), \quad (t,x) \in [0,+\infty[\times \mathbb{T}$$

$$B = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix},\ D + D^* \text{ positive-definite ;} \quad A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},\ A_{11} = A_{11}^*.$$

Coupling between parabolic and transport equations

$$f = \begin{pmatrix} f_h \\ f_p \end{pmatrix},\ \begin{cases} (\partial_t + A_{11}\partial_x + K_{11})f_h(t,x) + (A_{12}\partial_x + K_{12})f_p(t,x) = \mathbf{1}_\omega u_h(t,x) \\ (\partial_t - D\partial_x^2 + A_{22}\partial_x + K_{22})f_p(t,x) + (A_{21}\partial_x + K_{21})f_h(t,x) = \mathbf{1}_\omega u_p(t,x) \end{cases}$$

Question

For every, $f_0 \in L^2(\mathbb{T}, \mathbb{C}^d)$ does there exist $u \in L^2([0,T] \times \omega, \mathbb{C}^m)$ such that $f(T,\cdot) = 0$?

Outline

The results

Theorem (Case $M = I$, Beauchard-K-Le Balc'h 2020)

ω an open interval of $\mathbb{T}$.

$$T^* = \frac{2\pi - \text{length}(\omega)}{\min_{\mu \in \text{Sp}(A_{11})} |\mu|}$$

Then

1. *the system is not null-controllable on ω in time $T < T^*$,*
2. *the system is null-controllable on ω in time $T > T^*$.*

Minimal time = minimal time for the transport equation
In the case

$$\partial_t f_h + A_{11} \partial_x f_h = u_h \mathbf{1}_\omega$$

Free solutions = sums of waves travelling at speed $\mu_k \in \text{Sp}(A_{11})$.

The equation:

$$\partial_t f(t,x) + A\partial_x f(t,x) - B\partial_x^2 f(t,x) + Kf(t,x) = M\mathbf{1}_\omega u(t,x), \quad (t,x) \in [0,+\infty[\times \mathbb{T}.$$

Theorem (Underactuated system (K-Lissy 2023))

Null-controllability of every $H^{4d(d-1)}$ initial condition in time $T > T^*$ iff

$$\forall n \in \mathbb{Z},\ \mathrm{Vect}\{(n^2B + inA + K)^i Mv, i \in \mathbb{N}, v \in \mathbb{C}^d\} = \mathbb{C}^d$$

Coupling condition

n-th Fourier component of the parabolic-transport system:

$$X_n'(t) + (n^2B + inA + K)X_n(t) = Mu_n(t)$$

Condition of the theorem $\Leftrightarrow$ the finite-dimensional system $X_n' + (n^2B + inA + K)X_n = Mu_n$ is controllable.

Navier-Stokes

ρ: fluid density. v: fluid velocity. $a, \gamma, \mu > 0$.

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = \mathbf{1}_\omega u_1(t,x) \text{ on } [0,T] \times \mathbb{T} \\ \rho(\partial_t v + v\partial_x v) + \partial_x(a\rho^\gamma) - \mu\partial_x^2 v = \mathbf{1}_\omega u_2(t,x) \text{ on } [0,T] \times \mathbb{T} \end{cases}$$

Linearization around a stationnary state $(\bar\rho, \bar v) \in \mathbb{R}_+^* \times \mathbb{R}^*$:

$$\begin{cases} \partial_t \rho + \bar v \partial_x \rho + \bar\rho \partial_x v = \mathbf{1}_\omega u_1(t,x) \text{ sur } [0,T] \times \mathbb{T} \\ \partial_t v + \bar v \partial_x v + a\bar\rho^{\gamma-2}\partial_x \rho - \frac{\mu}{\bar\rho}\partial_x^2 v = \mathbf{1}_\omega u_2(t,x) \text{ on } [0,T] \times \mathbb{T} \end{cases}$$

Navier-Stokes

ρ: fluid density. v: fluid velocity. $a, \gamma, \mu > 0$.

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = \mathbf{1}_\omega u_1(t,x) \text{ on } [0,T] \times \mathbb{T} \\ \rho(\partial_t v + v \partial_x v) + \partial_x(a\rho^\gamma) - \mu \partial_x^2 v = \mathbf{1}_\omega u_2(t,x) \text{ on } [0,T] \times \mathbb{T} \end{cases}$$

Linearization around a stationnary state $(\bar\rho, \bar v) \in \mathbb{R}_+^* \times \mathbb{R}^*$:

$$\begin{cases} \partial_t \rho + \bar v \partial_x \rho + \bar\rho \partial_x v = \mathbf{1}_\omega u_1(t,x) \text{ sur } [0,T] \times \mathbb{T} \\ \partial_t v + \bar v \partial_x v + a\bar\rho^{\gamma-2} \partial_x \rho - \frac{\mu}{\bar\rho} \partial_x^2 v = \mathbf{1}_\omega u_2(t,x) \text{ on } [0,T] \times \mathbb{T} \end{cases}$$

- [Ervedoza-Guerrero-Glass-Puel 2012]: equation posed on $(0,L)$, boundary control acting on (ρ, v) in time $T > L/|\bar v|$
- [Chowdhury-Mitra-Ramaswamy-Renardy 2014]: velocity control in time $T > 2\pi/|\bar v|$ for the initial conditions $(\rho_0, v_0) \in H^1 \times L^2$.
- [K-Lissy 2023] with $A = \begin{pmatrix} \bar v & \bar\rho \\ a\bar\rho^{\gamma-2} & \bar v \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & \mu/\rho \end{pmatrix}$: velocity control, in time $T > (2\pi - \text{length}(\omega))/|\bar v|$ for initial conditions in $H^1 \times L^2$.

(Idea of the) proof: fully actuated system

Fourier components

$$(-B\partial_x^2 + A\partial_x + K)Xe^{inx} = n^2\left(B + \frac{i}{n}A - \frac{1}{n^2}K\right)Xe^{inx}$$

Spectrum of $-B\partial_x^2 + A\partial_x + K$

$$\mathsf{Sp}(-B\partial_x^2 + A\partial_x + K) = \left\{n^2\,\mathsf{Sp}\left(B + \frac{i}{n}A - \frac{1}{n^2}K\right)\right\}$$

Fourier components

$$(-B\partial_x^2 + A\partial_x + K)Xe^{inx} = n^2\left(B + \frac{i}{n}A - \frac{1}{n^2}K\right)Xe^{inx}$$

Spectrum of $-B\partial_x^2 + A\partial_x + K$

$$\mathrm{Sp}(-B\partial_x^2 + A\partial_x + K) = \left\{n^2\,\mathrm{Sp}\left(B + \frac{i}{n}A - \frac{1}{n^2}K\right)\right\}$$

Perturbation theory

λ_{nk} eigenvalue of $B + \frac{i}{n}A - \frac{1}{n^2}K$. λ_k eigenvalue of B: $\lambda_{nk} \to \lambda_k \in \mathrm{Sp}(B)$

- If $\lambda_k \neq 0$, $n^2\lambda_{nk} \underset{n\to+\infty}{\sim} n^2\lambda_k$: parabolic frequencies
- If $\lambda_k = 0$, $n^2\lambda_{nk} \underset{n\to+\infty}{\sim} in\mu_k$: hyperbolic frequencies

Fourier components

$$(-B\partial_x^2 + A\partial_x + K)Xe^{inx} = n^2\left(B + \frac{i}{n}A - \frac{1}{n^2}K\right)Xe^{inx}$$

Spectrum of $-B\partial_x^2 + A\partial_x + K$

$$\mathrm{Sp}(-B\partial_x^2 + A\partial_x + K) = \left\{n^2\,\mathrm{Sp}\left(B + \frac{i}{n}A - \frac{1}{n^2}K\right)\right\}$$

Perturbation theory

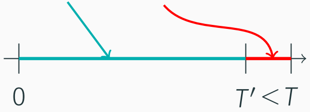
λ_{nk} eigenvalue of $B + \frac{i}{n}A - \frac{1}{n^2}K$. λ_k eigenvalue of B: $\lambda_{nk} \to \lambda_k \in \mathrm{Sp}(B)$

- If $\lambda_k \neq 0$, $n^2\lambda_{nk} \underset{n\to+\infty}{\sim} n^2\lambda_k$: parabolic frequencies
- If $\lambda_k = 0$, $n^2\lambda_{nk} \underset{n\to+\infty}{\sim} in\mu_k$: hyperbolic frequencies
- Free solutions: $= \sum X_{nk}e^{inx-n^2\lambda_{nk}t} \approx \sum_{\text{parabolic}} X_{nk}e^{inx-n^2\lambda_k t} + \sum_{\text{hyperbolic}} X_{nk}e^{inx-in\mu_k t}$
- Well-posed if $\Re(\lambda_k) > 0$ and $\mu_k \in \mathbb{R}$
- Not null-controllable in small time

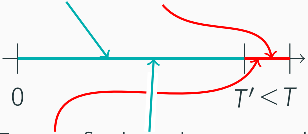
Decouple and control

- 

Decouple and control

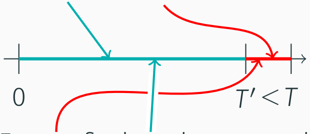
- For u_h, find u_p that controls parabolic frequencies in time T
- 

Decouple and control

- For u_h, find u_p that controls parabolic frequencies in time T
- 

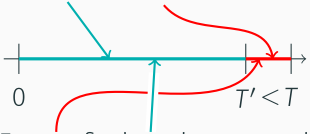
- For u_p, find u_h that controls the hyperbolic frequencies in time T

Decouple and control

- For u_h, find u_p that controls parabolic frequencies in time T
- 

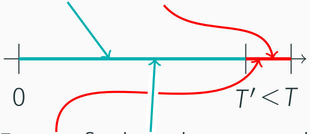
- For u_p, find u_h that controls the hyperbolic frequencies in time T
- If both steps agree, OK
- Make the two steps agree by choosing u_p smooth and using the Fredholm alternative (on a finite codimension subspace)

Decouple and control

- For u_h, find u_p that controls parabolic frequencies in time T
- 

- For u_p, find u_h that controls the hyperbolic frequencies in time T
- If both steps agree, OK
- Make the two steps agree by choosing u_p smooth and using the Fredholm alternative (on a finite codimension subspace)
- Step 1: null-controllability of a parabolic equation in time $T - T' > 0$
- Step 2: exact controllability of a perturbed transport equation in time T'. Ok if $T' > T^*$.

Decouple and control

- For u_h, find u_p that controls parabolic frequencies in time T
- 

- For u_p, find u_h that controls the hyperbolic frequencies in time T
- If both steps agree, OK
- Make the two steps agree by choosing u_p smooth and using the Fredholm alternative (on a finite codimension subspace)
- Step 1: null-controllability of a parabolic equation in time $T - T' > 0$
- Step 2: exact controllability of a perturbed transport equation in time T'. Ok if $T' > T^*$.
- Deal the finite dimensional subspaces that are left: compactness-uniqueness

(Idea of the) proof: underactuated systems

Theorem

If $\text{Vect}\{B^i M X, i \in \mathbb{N}, X \in \mathbb{C}^d\} = \mathbb{C}^d$*, for every* X_0*,* $X_T \in \mathbb{C}^d$*, there exists* $u \in H^k_0(0,T)$ *such that*

$$X' = BX + Mu, \quad X(0) = X_0$$

satisfies $X(T) = X_T$*.*

Theorem

If $\mathrm{Vect}\{B^i M X, i \in \mathbb{N}, X \in \mathbb{C}^d\} = \mathbb{C}^d$, *for every* X_0, $X_T \in \mathbb{C}^d$, *there exists* $u \in H_0^k(0,T)$ *such that*

$$X' = BX + Mu, \quad X(0) = X_0$$

satisfies $X(T) = X_T$.

Proof through algebraic solvability ($X_0 = 0$).

Case $M = I$. With $X(t) = \frac{t}{T} X_T$: $X' = \frac{1}{T} X_T$

Theorem

If Vect$\{B^i M X, i \in \mathbb{N}, X \in \mathbb{C}^d\} = \mathbb{C}^d$, *for every* X_0, $X_T \in \mathbb{C}^d$, *there exists* $u \in H_0^k(0,T)$ *such that*

$$X' = BX + Mu, \quad X(0) = X_0$$

satisfies $X(T) = X_T$.

Proof through algebraic solvability ($X_0 = 0$).

Case $M = I$. With $X(t) = \frac{t}{T} X_T$: $\quad X' = BX + \underbrace{\frac{1}{T} X_T - \frac{t}{T} B X_T}_{u(t)}$

Theorem

If $\mathrm{Vect}\{B^i M X, i \in \mathbb{N}, X \in \mathbb{C}^d\} = \mathbb{C}^d$, *for every* $X_0, X_T \in \mathbb{C}^d$, *there exists* $u \in H^k_0(0,T)$ *such that*

$$X' = BX + Mu, \quad X(0) = X_0$$

satisfies $X(T) = X_T$.

Proof through algebraic solvability ($X_0 = 0$).

Case $M = I$. With $X(t) = \frac{t}{T} X_T$: $\quad X' = BX + \underbrace{\frac{1}{T} X_T - \frac{t}{T} B X_T}_{u(t)}$

Exercise: add suitable $Y(t) \in C^\infty([0,T])$ with $Y(0) = Y(T) = 0$ to X so that

$$(X+Y)' = B(X+Y) + \underbrace{u(t) + Y'(t) + BY(t)}_{\tilde{u}(t)}$$

with $\tilde{u} \in H^k_0(0,T)$.

Theorem

If Vect$\{B^i MX, i \in \mathbb{N}, X \in \mathbb{C}^d\} = \mathbb{C}^d$, *for every* X_0, $X_T \in \mathbb{C}^d$, *there exists* $u \in H_0^k(0,T)$ *such that*

$$X' = BX + Mu, \quad X(0) = X_0$$

satisfies $X(T) = X_T$.

Proof through algebraic solvability ($X_0 = 0$).
General M. Kalman matrix: $[B|M] := \begin{pmatrix} M & BM & \cdots & B^{d-1}M \end{pmatrix}$. Rank$([B|M]) = d$.

Let $w = (w_1, \ldots, w_{d-1}) \in H_0^{d-1}$ such that $Y' = BY + \underbrace{[B|M]w}_{\text{Control for the case } M = I}$, $Y(0) = 0$, $Y(T) = X_T$.

Theorem

If $\mathrm{Vect}\{B^i M X, i \in \mathbb{N}, X \in \mathbb{C}^d\} = \mathbb{C}^d$*, for every* $X_0, X_T \in \mathbb{C}^d$*, there exists* $u \in H_0^k(0,T)$ *such that*

$$X' = BX + Mu, \quad X(0) = X_0$$

satisfies $X(T) = X_T$*.*

Proof through algebraic solvability ($X_0 = 0$).

General M*.* Kalman matrix: $[B|M] := \begin{pmatrix} M & BM & \cdots & B^{d-1}M \end{pmatrix}$. Rank$([B|M]) = d$.

Let $w = (w_1, \dots, w_{d-1}) \in H_0^{d-1}$ such that $Y' = BY + \underbrace{[B|M]w}_{\text{Control for the case } M = I}$, $Y(0) = 0$, $Y(T) = X_T$.

Goal: with $u := w_1 + w_2' + \cdots + w_d^{(d-1)}$, $X' = BX + Mu$, $X(0) = 0$, $X(T) = X_T$.

$$\overbrace{\begin{pmatrix} \underbrace{\partial_t - B}_{\text{Operator on } L^2(\mathbb{T})^d} & M \end{pmatrix}}^{\mathcal{P}} \overbrace{\begin{pmatrix} 0 & -M & \cdots & -\sum_{j=0}^{d-2} \partial_t^j B^{d-2-j} M \\ I & \partial_t & \cdots & \partial_t^{d-1} \end{pmatrix}}^{\mathcal{M}} = [B|M]$$

$$Y' - BY = \mathcal{P} \circ \mathcal{M}(w) = (\partial_t - B)\mathcal{M}_1 w + M\mathcal{M}_2 w = (\partial_t - B)\mathcal{M}_1 w + Mu \qquad \square$$

Theorem (Underactuated system (K-Lissy 2023))

Null-controllability of every $H^{4d(d-1)}$ initial condition in time $T > T^*$ if

$$\forall n \in \mathbb{Z},\ \mathrm{Rank}([B_n|M]) = d.$$

Algebraic solvability on each Fourier components?

$$(\partial_t - B\partial_x^2 + A\partial_x + K)f = \mathbf{1}_\omega v$$

$$\xrightarrow{\text{Fourier}} X_n' = B_n X_n + v_n$$

$$\xrightarrow{\text{Kalman condition}} X_n' = B_n X_n + [B_n|M]w_n$$

$$\xrightarrow{\text{Algebraic Solvability}} X_n' = B_n X_n + M u_n$$

$$\xrightarrow{\text{Inverse Fourier}} (\partial_t - B\partial_x^2 + A\partial_x + K)f = Mu$$

Theorem (Underactuated system (K-Lissy 2023))

Null-controllability of every $H^{4d(d-1)}$ initial condition in time $T > T^*$ if

$$\forall n \in \mathbb{Z}, \ \text{Rank}([B_n|M]) = d.$$

Algebraic solvability on each Fourier components?

$$(\partial_t - B\partial_x^2 + A\partial_x + K)f = \mathbf{1}_\omega v$$

$$\xrightarrow{\text{Fourier}} X_n' = B_n X_n + v_n$$

$$\xrightarrow{\text{Kalman condition}} X_n' = B_n X_n + [B_n|M]w_n$$

$$\xrightarrow{\text{Algebraic Solvability}} X_n' = B_n X_n + M u_n$$

$$\xrightarrow{\text{Inverse Fourier}} (\partial_t - B\partial_x^2 + A\partial_x + K)f = Mu$$

$u = R(\partial_t, \partial_x)v$ with $R(\tau, n) = P(\tau, n)/Q(n)$ (rational function): no guarentee on $\text{Supp}(u)$

Theorem (Underactuated system (K-Lissy 2023))

Null-controllability of every $H^{4d(d-1)}$ initial condition in time $T > T^*$ if

$$\forall n \in \mathbb{Z},\ \text{Rank}([B_n|M]) = d.$$

Algebraic solvability on each Fourier components?

$$(\partial_t - B\partial_x^2 + A\partial_x + K)f = \mathbf{1}_\omega Q(\partial_x)v \quad (v \text{ controls } Q(\partial_x)^{-1}f_0)$$

$$\xrightarrow{\text{Fourier}} X_n' = B_n X_n + Q(-in)v_n$$

$$\xrightarrow{\text{Kalman condition}} X_n' = B_n X_n + [B_n|M]Q(-in)[B_n|M]^{-1}v_n$$

$$\xrightarrow{\text{Algebraic Solvability}} X_n' = B_n X_n + MP(\partial_t, -in)v_n$$

$$\xrightarrow{\text{Inverse Fourier}} (\partial_t - B\partial_x^2 + A\partial_x + K)f = MP(\partial_t, \partial_x)v$$

$u = R(\partial_t, \partial_x)Q(\partial_x)v$ with $R(\tau, n) = P(\tau, n)/Q(n)$ (rational function):
$\text{Supp}(u) \subset \text{Supp}(v)$

Some refinements

Loss of regularity

- Null-controllability of every $H^{4d(d-1)}(\mathbb{T})^d$ initial condition: very crude regularity assumption
- Better regularity assumption: make the computations, hope that you find an L^2 control
- *Some* regularity assumption is needed in general
- $\begin{cases} (\partial_t + \partial_x)f_{\mathrm{h}} + \partial_x f_{\mathrm{p}} + f_{\mathrm{p}} = 0 \\ (\partial_t - \partial_x^2)f_{\mathrm{p}} = \mathbb{1}_\omega u_{\mathrm{p}} \end{cases}$
  Smoothing: if $f_{0,\mathrm{h}} \notin H^1$, we cannot steer f_0 to 0 with L^2 controls

Equations with invariants

$$\begin{cases} (\partial_t + \partial_x) f_h + \partial_x f_p = 0 \\ (\partial_t - \partial_x^2) f_p = \mathbf{1}_\omega u_p \end{cases}$$ not null-controllable:

for $n = 0$, $\mathrm{Vect}\{(n^2 B + inA + K)^i M v, i \in \mathbb{N}, v \in \mathbb{C}^d\} = \mathrm{Vect}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \neq \mathbb{C}^d$

The average of the hyperbolic component is conserved. Maybe null-controllability of every initial condition with zero hyperbolic-average?

Equations with invariants

$$\begin{cases} (\partial_t + \partial_x) f_h + \partial_x f_p = 0 \\ (\partial_t - \partial_x^2) f_p = \mathbf{1}_\omega u_p \end{cases}$$ not null-controllable:

for $n = 0$, $\mathrm{Vect}\{(n^2 B + inA + K)^i M v, i \in \mathbb{N}, v \in \mathbb{C}^d\} = \mathrm{Vect}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \neq \mathbb{C}^d$

The average of the hyperbolic component is conserved. Maybe null-controllability of every initial condition with zero hyperbolic-average?

Theorem ((K-Lissy 2023))

Assume $T > T_*$ and

- $\forall |n|$ *large enough*, $\mathrm{Vect}\{(n^2 B + inA + K)^i M v, i \in \mathbb{N}, v \in \mathbb{C}^d\} = \mathbb{C}^d$
- $f_0 \in H^{4d(d-1)}(\mathbb{T})^d$
- $\forall n \in \mathbb{Z}$, $\widehat{f_0}(n) \in \mathrm{Vect}\{(n^2 B + inA + K)^i M v, i \in \mathbb{N}, v \in \mathbb{C}^d\}$

There exists a control in $L^2((0,T) \times \omega)$ that steers f_0 to 0 in time T.

Conclusion

Open problems

- domain other than $\mathbb{T}$?
- non-constant coefficients?
- unique continuation?
- …

That’s all folks!