NULL-CONTROLLABILITY FOR WEAKLY DISSIPATIVE HEAT-LIKE EQUATIONS

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ABSTRACT. We study the null-controllability properties of heat-like equations posed on the whole Euclidean space $\mathbb{R}^n$. These evolution equations are associated with Fourier multipliers of the form $\rho(|D_x|)$, where $\rho : [0,+\infty) \to \mathbb{C}$ is a measurable function such that $\operatorname{Re}\rho$ is bounded from below. We consider the "weakly dissipative" case, a typical example of which is given by the fractional heat equations associated with the multipliers $\rho(\xi) = \xi^s$ in the regime $s \in (0,1)$, for which very few results exist. We identify sufficient conditions and necessary conditions on the control supports for the null-controllability to hold. More precisely, we prove that these equations are null-controllable in any positive time from control supports which are sufficiently thick at all scales. Under assumptions on the multiplier ρ, in particular assuming that $\rho(\xi) = o(\xi)$, we also prove that the null-controllability implies that the control support is thick at all scales, with an explicit lower bound of the thickness ratio in terms of the multiplier ρ. Finally, using Smith-Volterra-Cantor sets, we provide examples of non-trivial control supports that satisfy these necessary or sufficient conditions.

1. Introduction

1.1. **Motivation.** We study the null-controllability properties of the following class of parabolic heat-like equations

$$(E_\rho)\qquad \begin{cases} \partial_t f(t,x) + \rho(|D_x|)f(t,x) = \mathbb{1}_\omega u(t,x), & (t,x) \in \mathbb{R}_+^* \times \mathbb{R}^n, \\ f(0,\cdot) = f_0 \in L^2(\mathbb{R}^n). \end{cases}$$

Above, the operator $\rho(|D_x|)$ is the Fourier multiplier associated with the symbol $\rho(|\xi|)$, with $|\cdot|$ the canonical Euclidean norm in $\mathbb{R}^n$, the function $\rho : [0,+\infty) \to \mathbb{C}$ being measurable such that $\operatorname{Re}\rho$ is bounded from below, and $\omega \subset \mathbb{R}^n$ is a measurable set with positive Lebesgue measure. We investigate the relationship between the geometry of ω and the null-controllability properties of these heat-like equations, defined as follows.

Definition 1. Let $T > 0$ and $\omega \subset \mathbb{R}^n$ be a measurable set with positive measure. The equation (E_ρ) is said to be *null-controllable* from ω in time $T > 0$ when for all $f_0 \in L^2(\mathbb{R}^n)$, there exists a control $u \in L^2((0,T)\times\omega)$ such that the mild solution of (E_ρ) satisfies $f(T,\cdot) = 0$.

Although the null-controllability properties of parabolic equations posed on bounded domains of $\mathbb{R}^n$ have been known for years [19, 22, 23, 3, 4, 7], the same study for parabolic equations posed on the whole Euclidean space $\mathbb{R}^n$, as the equations (E_ρ), is quite recent. It follows from previous works [1, 2, 6, 11, 12, 14, 21, 25, 26] that the null-controllability properties of such models, and also their approximate null-controllability or the stabilization properties, are associated with the geometric notion of thickness, defined as follows.

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Definition 2. Given $\gamma \in (0,1)$ and $r > 0$, the set $\omega \subset \mathbb{R}^n$ is said to be γ-*thick* at scale r, or (γ, r)-thick, when it is measurable and satisfies

$$\forall x \in \mathbb{R}^n, \quad \operatorname{Leb}(\omega \cap B(x,r)) \geq \gamma \operatorname{Leb}(B(x,r)),$$

where Leb denotes the Lebesgue measure in $\mathbb{R}^n$.

Precisely, it is known from the work [2] that the thickness is a geometric necessary condition for the null-controllability of the general equations (E_ρ) (in fact, more generally, for the rapid stabilization of such equations). This condition also turns out to be a necessary and sufficient condition that ensures the null-controllability in any positive time $T > 0$ of the fractional heat equations (E_{ρ_s}) associated with the multipliers $\rho_s(t) = t^s$ in the regime $s > 1$ [1, 12, 24, 26]. For this particular class of equations, which will serve as a common thread throughout this introduction, it was also proven in [16] that in the weak-dissipation regime $s \in (0,1)$, positive null-controllability results can not be obtained from control supports $\omega \subset \mathbb{R}^n$ which are not dense in the whole space $\mathbb{R}^n$, this result being also established in the critical dissipation regime $s = 1$ (corresponding to the half-heat equation) in dimension $n = 1$ [17, Théorème 2.3][20].

From this point on, two areas of work naturally emerge. On the one hand, it would be interesting to characterize the multipliers ρ for which the thickness is a necessary and sufficient geometric condition that ensures the null-controllability of the associated evolution equations (E_ρ), as with the strongly-dissipative heat equations (it is worth noting that the generalized Lebeau-Robianno's method as stated by Duyckaerts and Miller [10, Theorem 6.1] together with Kovrijkine's spectral estimate [18] gives a first result on this topic). One the other hand, it would be interesting to study the null-controllability properties of the heat-like equations (E_ρ) in a weak-dissipation setting, for which very few results have been obtained so far, and to continue the studies carried out in the works [16, 17, 20], in particular by looking for control supports from which these equations can be controlled to zero. In the following, we will only focus on this second point.

1.2. **Main results.** In the present work, we prove that the null-controllability properties of the parabolic heat-like equations (E_ρ) is associated with the following stronger notion of thickness in the weak-dissipation regime.

Definition 3. Given some $r_0 > 0$ and a function $\gamma : (0, r_0] \to [0,1]$, a measurable set $\omega \subset \mathbb{R}^n$ is said to be *thick with respect to* γ (or γ-*thick*) when it satisfies that for every $r \in (0, r_0]$ and $x \in \mathbb{R}^n$,

$$\operatorname{Leb}(\omega \cap B(x,r)) \geq \gamma(r) \operatorname{Leb}(B(x,r)).$$

This definition can be rephrased as "ω is $\gamma(r)$-thick at every scale $r \in (0, r_0]$". Therefore, being thick with respect to γ is a far stronger notion than the usual notion of thickness.

We first prove a general result stating that the parabolic equation (E_ρ) is always null-controllable from control supports $\omega \subset \mathbb{R}^n$ being thick with respect to some function γ_ρ associated to the multiplier ρ (under reasonnable assumptions).

Theorem 4. *Let $\rho : [0, +\infty) \to \mathbb{C}$ be a function such that $\operatorname{Re}\rho$ is a non-negative continuous function satisfying $\lim_{+\infty} \operatorname{Re}\rho = +\infty$. Let $r_0 > 0$ and $\gamma_\rho : (0, r_0] \to (0,1]$ be the function defined by*

$$\gamma_\rho(r) := c_0 \exp(-c_1 (\operatorname{Re}\rho)(1/r)^\alpha), \tag{1}$$

where $c_0 \in (0,1)$, $c_1 > 0$ and $\alpha \in (0,1)$ are some parameters. Let $\omega \subset \mathbb{R}^n$ be γ_ρ-thick. Then, for every $T > 0$, the parabolic equation (E_ρ) is null-controllable from ω in time T.

Under additional assumptions on the multiplier ρ, we also prove that the thickness with respect to some function $\tilde{\gamma}_\rho : (0, r_0] \to [0,1]$ is also a necessary condition for the null-controllability to hold.

Theorem 5. *Let $K > 0$ and $\mathcal{C} = \{\xi \in \mathbb{C}, \operatorname{Re}(\xi) > K, |\operatorname{Im}(\xi)| < K^{-1} \operatorname{Re}(\xi)\}$. Let $\rho : \mathcal{C} \cup \mathbb{R}_+ \to \mathbb{C}$ be such that*

- *ρ is holomorphic on $\mathcal{C}$,*
- *$\rho(\xi) = o(\xi)$ as $|\xi| \to \infty$, $\xi \in \mathcal{C}$,*
- *ρ is measurable on $\mathbb{R}_+$ and $\inf_{\mathbb{R}_+} \mathrm{Re}(\rho) > -\infty$,*
- *there exists $C > 0$ such that for $\xi \in \mathcal{C}$, $|\mathrm{Im}\,\rho(\xi)| \leq C\,\mathrm{Re}\,\rho(\xi)$,*
- *$\ln(\xi) = o(\mathrm{Re}\,\rho(\xi))$ in the limit $|\xi| \to +\infty$, $\xi \in \mathcal{C}$.*

Let $T > 0$ and $\omega \subset \mathbb{R}^n$ be measurable. Assume that the parabolic equation (E_ρ) is null controllable from ω in time $T > 0$.

There exists $\lambda > 0$, $r_0 > 0$ and $c > 0$ such that for every $\epsilon > 0$, and for every function $r \in (0, r_0] \mapsto h_r \in \mathbb{R}_+^*$ that satisfy

$$\forall r \in (0, r_0], \sqrt{h_r 2T(1+\epsilon)\,\mathrm{Re}\,\rho\left(\frac{\lambda}{h_r}\right)} \leq r, \tag{2}$$

then, there exists $r_1 \in (0, r_0)$ such that for every $0 < r < r_1$, and $x \in \mathbb{R}^n$, we have

$$\frac{\mathrm{Leb}(\omega \cap B(x,r))}{\mathrm{Leb}(B(x,r))} \geq c r^{-n} \exp\left(-2T(1+\epsilon)\,\mathrm{Re}\,\rho\left(\frac{\lambda}{h_r}\right)\right).$$

Remark 6.

- The conclusion of this theorem can be rephrased as "ω is γ-thick", where
$$\gamma : (0, r_1] \ni r \mapsto c r^{-n} \exp\left(-2T(1+\epsilon)\,\mathrm{Re}\,\rho\left(\frac{\lambda}{h_r}\right)\right).$$
- The hypothesis $\rho(\xi) = o(\xi)$ is a rigorous way of saying that the heat-like equation (E_ρ) is weakly dissipative. Since the null-controllability is known to hold for strongly dissipative equations (e.g., the heat equation) on any thick set, one cannot expect to obtain theorem 5 without a hypothesis of this kind.
- The hypotheses $|\mathrm{Im}\,\rho(\xi)| \leq C\,\mathrm{Re}\,\rho(\xi)$ and $\ln(\xi) = o(\mathrm{Re}\,\rho(\xi))$ are mainly assumed for cosmetic reasons. We could prove some results with weaker hypotheses, but the result (and the proof) would be even more tedious. This would be of dubious interest, as such, we prefer not to detail these results here.
- The holomorphy hypothesis is a technical limitation of our strategy of proof involving complex deformation of integration path. Proving a version of theorem 5 without this hypothesis is an open problem.

Remark 7. Given some function $\gamma : (0, r_0] \to [0,1]$, an example of set being γ-thick is of course the whole space $\mathbb{R}^n$, but it might be difficult to visualize non trivial examples of sets satisfying this property. In section 4, we construct subsets ω of $\mathbb{R}^n$ which are γ-thick and such that $\mathrm{Leb}(\mathbb{R}^n \setminus \omega) > 0$. Roughly speaking, when γ is assumed to be decreasing and satisfying $\gamma(r) \to 0$ as $r \to 0$, these are complements of Smith-Volterra-Cantor sets associated with the sequence $\left(24(\gamma(2^{-n}) - \gamma(2^{-n-1}))\right)_{n\geq 0}$ (whose definition we recall in definition 17). We refer to proposition 20 for the details.

Let us now apply theorem 4 and theorem 5 to the fractional heat equations.

Example 8. For all positive real number $s > 0$, let us consider the multiplier $\rho_s : [0, +\infty) \to [0, +\infty)$ defined for all $t \geq 0$ by $\rho_s(t) = t^s$. Let us consider a positive time $T > 0$ and a measurable set $\omega \subset \mathbb{R}^n$. As recalled in the beginning of the introduction, the null-controllability properties of the associated fractional heat equation (E_{ρ_s}) are well understood in the high-dissipation regime $s > 1$. We will therefore only focus on the weak-dissipation regime $s \in (0,1]$.

On the one hand, it follows from theorem 4 that in the regime $s \in (0,1]$, and when the set ω is thick with respect to the function $\gamma_s(r) = c_0 \exp(-c_1 r^{-\alpha s})$, where $c_0, c_1 > 0$ and $\alpha \in (0,1)$ are parameters, then the fractional heat equation (E_{ρ_s}) is null-controllable from ω at time T. As far as we know, this is the first positive null-controllability result for the fractional heat equation in the weak-dissipative regime $s \in (0,1]$.

On the other hand, notice that when $s \in (0,1)$, the condition (17) of theorem 5 is satisfied with $h_r = r^{2/(1-s)}$ for the above multiplier ρ_s. As a consequence, still in the regime $s \in (0,1)$, it follows from theorem 5 that if the fractional heat equation (E_{ρ_s}) is null-controllable from ω at time T, then there exist some positive constants $c_0, c_1 > 0$ such that ω is thick with respect to the function $\gamma_s(r) = c_0 \exp(-c_1 r^{-2s/(1-s)})$. Notice that we do not consider the critical case $s = 1$, whose understanding remains an open problem.

Remark 9. Let us consider the fractional heat equation associated to the Fourier multiplier $\rho_s(t) = t^s$ as above, with $s \leq 1$. In dimension one, one popular way to study the null-controllability of PDEs is the *moment method*. The is the strategy employed by Micu and Zuazua [22], using *shaped controls*, i.e., controls of the form $u(t)h(x)$.

Our results underline the difference between *shaped controls* and *internal controls* (the kind of controls we are considering). Indeed, example 8 show that if ω is sufficiently thick, the fractional heat equation is null controllable with internal controls; but if we consider *shaped controls*, then null-controllability *never* holds, whatever the profile h is [22, Section 5] (see also [23, Appendix]).

2. Sufficient condition

This section is devoted to the proof of theorem 4, which states that given a function $\rho : [0, +\infty) \to \mathbb{C}$ such that $\operatorname{Re}\rho$ is a continuous non-negative function such that $\lim_{+\infty} \operatorname{Re}\rho = +\infty$, the parabolic equation (E_ρ) is null-controllable from any set $\omega \subset \mathbb{R}^n$ being thick with respect to the density $\gamma_\rho : (0, r_0] \to (0,1]$ defined by eq. (1), and in any positive time $T > 0$.

By the Hilbert Uniqueness Method (see, e.g., [9, Theorem 2.44]), the null-controllability of the parabolic equation (E_ρ) is equivalent to the observability of the heat-like semigroup $(\mathrm{e}^{-t\overline{\rho}(|D_x|)})_{t\geq 0}$, that we recall in the following definition.

Definition 10. Let $T > 0$, and let $\omega \subset \mathbb{R}^n$ be measurable. The semigroup $(\mathrm{e}^{-t\overline{\rho}(|D_x|)})_{t\geq 0}$ is said to be observable from the set ω in time T if there exists a positive constant $C_{\omega,T} > 0$ such that for all $g \in L^2(\mathbb{R}^n)$,

$$\|\mathrm{e}^{-T\overline{\rho}(|D_x|)} g\|^2_{L^2(\mathbb{R}^n)} \leq C_{\omega,T} \int_0^T \|\mathrm{e}^{-t\overline{\rho}(|D_x|)} g\|^2_{L^2(\omega)} \, \mathrm{d}t.$$

We will prove that the heat-like semigroup $(\mathrm{e}^{-t\overline{\rho}(|D_x|)})_{t\geq 0}$ is indeed observable, with an upper bound on the observability constant. This will imply that the heat-like equation (E_ρ) is null-controllable.

Theorem 11. *Let $c_0 \in (0,1)$, $c_1 > 0$, $\alpha \in (0,1)$ and let γ_ρ be defined by eq. (1). Let $\omega \subset \mathbb{R}^n$ be γ_ρ-thick. There exists a positive constant $C > 0$ such that for all $T > 0$ and $g \in L^2(\mathbb{R}^n)$,*

$$\|\mathrm{e}^{-T\overline{\rho}(|D_x|)} g\|^2_{L^2(\mathbb{R}^n)} \leq C \exp\left(\frac{C}{T^{\alpha/(1-\alpha)}}\right) \int_0^T \|\mathrm{e}^{-t\overline{\rho}(|D_x|)} g\|^2_{L^2(\omega)} \, \mathrm{d}t,$$

where $\alpha \in (0,1)$ is the parameter appearing in the definition of the function γ_ρ.

The rest of this section is devoted to the proof of theorem 11. This will be done in two steps: first proving a spectral estimate reminiscent of Jerison and Lebeau's spectral inequality [15, Theorem 14.6] or Logvinenko-Sereda-Kovrijkine estimate [18], and second using Lebeau and Robbiano's method, as stated in the following theorem proven by Beauchard and Pravda-Starov.

Theorem 12 (Theorem 2.1 in [5]). *Let ω be a measurable subset of $\mathbb{R}^n$ with positive Lebesgue measure, $(\pi_k)_{k\geq 1}$ be a family of orthogonal projections defined on $L^2(\mathbb{R}^n)$ and $(e^{tA})_{t\geq 0}$ be a contraction semigroup* *on $L^2(\mathbb{R}^n)$. Assume that there exist $c_1, c_2, a, b, t_0, m > 0$ some positive constants with $a < b$ such that the following spectral inequality*

$$\forall g \in L^2(\mathbb{R}^n), \forall k \geq 1, \quad \|\pi_k g\|_{L^2(\mathbb{R}^n)} \leq e^{c_1 k^a} \|\pi_k g\|_{L^2(\omega)},$$

and the following dissipation estimate

$$\forall g \in L^2(\mathbb{R}^n), \forall k \geq 1, \forall 0 < t < t_0, \quad \|(1-\pi_k)(e^{tA} g)\|_{L^2(\mathbb{R}^n)} \leq \frac{1}{c_2} e^{-c_2 t^m k^b} \|g\|_{L^2(\mathbb{R}^n)},$$

hold. Then, there exists a positive constant $C > 1$ such that the following observability estimate holds

$$\forall T > 0, \forall g \in L^2(\mathbb{R}^n), \quad \|e^{TA} g\|^2_{L^2(\mathbb{R}^n)} \leq C \exp\left(\frac{C}{T^{\frac{am}{b-a}}}\right) \int_0^T \|e^{tA} g\|^2_{L^2(\omega)} \, \mathrm{d}t.$$

In the rest of this section, in order to alleviate the text, we will denote the spectral projectors associated with the operator $(\operatorname{Re}\rho)(|D_x|)$ as follows

$$\pi_{\lambda,\rho} = \mathbb{1}_{(-\infty,\lambda]}((\operatorname{Re}\rho)(|D_x|)), \quad \lambda \geq 0. \tag{3}$$

Let us now prove the following spectral estimates.

Proposition 13. *Let $c_0 \in (0,1)$, $c_1 > 0$, $\alpha \in (0,1)$ and let γ_ρ be defined by eq. (1). Let $\omega \subset \mathbb{R}^n$ be γ_ρ-thick. Then, there exists a positive constant $c > 0$ such that*

$$\forall \lambda > 0, \forall g \in L^2(\mathbb{R}^n), \quad \|\pi_{\lambda,\rho} g\|_{L^2(\mathbb{R}^n)} \leq c e^{c\lambda^\alpha} \|\pi_{\lambda,\rho} g\|_{L^2(\omega)}.$$

Remark 14. The proof of proposition 13 will be based on Kovrikine's spectral estimate [18, Theorem 3] stating that there exists a universal positive constant $K > 0$ depending only on the dimension n such that for all (γ, L)-thick set $\omega \subset \mathbb{R}^n$, with $\gamma \in (0,1]$ and $L > 0$, for all $\lambda \geq 0$ and $g \in L^2(\mathbb{R}^n)$ such that $\operatorname{Supp} \widehat{g} \subset B(0,\lambda)$, we have

$$\|g\|_{L^2(\mathbb{R}^n)} \leq \left(\frac{K}{\gamma}\right)^{K(1+L\lambda)} \|g\|_{L^2(\omega)}. \tag{4}$$

Proof of proposition 13. Let us consider some $\lambda > 0$ and $g \in L^2(\mathbb{R}^n)$ be fixed. Recall that by definition of thickness with respect to γ_ρ, the set ω is $\gamma_\rho(r)$-thick at every scale $r \in (0, r_0]$, meaning that the following estimate holds for every $r \in (0, r_0]$ and $x \in \mathbb{R}^n$,

$$\operatorname{Leb}(\omega \cap B(x,r)) \geq \gamma(r) \operatorname{Leb}(B(x,r)).$$

On the other hand, by definition (3) of the spectral projector $\pi_{\lambda,\rho}$, the function $\widehat{\pi_{\lambda,\rho} g}$ is supported in $B(0, \rho^\dagger(\lambda))$, where

$$\rho^\dagger(\lambda) := \sup\{\mu \geq 0 : \operatorname{Re}\rho(\mu) \leq \lambda\}.$$

Notice that $\rho^\dagger(\lambda)$ is well-defined since $\lim_{+\infty} \operatorname{Re}\rho = +\infty$ by assumption. We therefore deduce from the spectral estimate (4) that for all $r \in (0, r_0]$,

$$\|\pi_{\lambda,\rho} g\|_{L^2(\mathbb{R}^n)} \leq \left(\frac{K}{\gamma_\rho(r)}\right)^{K(1+r\rho^\dagger(\lambda))} \|\pi_{\lambda,\rho} g\|_{L^2(\omega)},$$

where the constant $K > 0$ only depends on the dimension n. Assume for now that $\lambda \geq \lambda_\rho$, where $\lambda_\rho > 0$ is defined so that

$$\forall \lambda \geq \lambda_\rho, \quad \rho^\dagger(\lambda) \geq 1/r_0.$$

Then, by choosing $r = 1/\rho^\dagger(\lambda) \in (0, r_0]$ in the above estimate, we obtain that

$$\|\pi_{\lambda,\rho} g\|_{L^2(\mathbb{R}^n)} \leq \left(\frac{K}{\gamma_\rho(1/\rho^\dagger(\lambda))}\right)^{2K} \|\pi_{\lambda,\rho} g\|_{L^2(\omega)}.$$

Moreover, since the function $\operatorname{Re}\rho$ is continuous, it follows from the definition of γ_ρ and the definition of $\rho^\dagger(\lambda)$ that

$$\gamma_\rho(1/\rho^\dagger(\lambda)) = c_0\exp(-c_1(\operatorname{Re}\rho)(\rho^\dagger(\lambda))^\alpha) = c_0\exp(-c_1\lambda^\alpha).$$

As consequence, we obtain the following estimate

$$\|\pi_{\lambda,\rho}g\|_{L^2(\mathbb{R}^n)} \le \left(\frac{K}{c_0}\right)^{2K} e^{2c_1K\lambda^\alpha}\|\pi_{\lambda,\rho}g\|_{L^2(\omega)}.$$

For the case $0<\lambda<\lambda_\rho$, we use again Kovrijkine's estimate to find a C_1 such that for every $g\in L^2(\mathbb{R}^n)$, $\|\pi_{\lambda,\rho}g\|_{L^2(\mathbb{R}^n)} \le C_1\|\pi_{\lambda,\rho}g\|_{L^2(\omega)}$, since $\pi_{\lambda,\rho}(L^2(\mathbb{R}^n))\subset\pi_{\lambda_\rho,\rho}(L^2(\mathbb{R}^n))$. Therefore, for $c_2=2c_1K$ and $C_2=\max(C_1,(K/c_0)^{2K})$, we have

$$\|\pi_{\lambda,\rho}g\|_{L^2(\mathbb{R}^n)} \le C_2\mathrm{e}^{c_2\lambda^\alpha}\|\pi_{\lambda,\rho}g\|_{L^2(\omega)}.$$

This ends the proof of proposition 13. □

Proof of theorem 11. Given a positive time $T>0$ and a set $\omega\subset\mathbb{R}^n$ being thick with respect to the function γ_ρ defined in (1), we are now in position to prove an observability estimate for the semigroup $(\mathrm{e}^{-t\overline{\rho}(|D_x|)})_{t\ge0}$ from ω in time T. First notice from Plancherel's theorem that the following dissipation estimates hold for any $t>0$, $k\ge1$ and $g\in L^2(\mathbb{R}^n)$,

$$\|(1-\pi_{k,\rho})(\mathrm{e}^{-t\overline{\rho}(|D_x|)}g)\|_{L^2(\mathbb{R}^n)} = \|(1-\pi_{k,\rho})(\mathrm{e}^{-t(\operatorname{Re}\rho)(|D_x|)}g)\|_{L^2(\mathbb{R}^n)} \le e^{-tk}\|g\|_{L^2(\mathbb{R}^n)}.$$

On the other hand, it follows from proposition 13 that there exists a positive constant $c>0$ such that

$$\forall k\ge1, \forall g\in L^2(\mathbb{R}^n),\quad \|\pi_{k,\rho}g\|_{L^2(\mathbb{R}^n)} \le c\mathrm{e}^{c\lambda^\alpha}\|\pi_{k,\rho}g\|_{L^2(\omega)},$$

where $\alpha\in(0,1)$ is the parameter appearing in the definition of the function γ_ρ. Theorem 11 is then a consequence of theorem 12. □

3. Necessary condition

The aim of this section is to prove theorem 5. To that end, we will use some asymptotics on the evolution of coherent states [16, Section 4], that we recall here.

Let $K>0$, and $\mathcal{C}\subset\mathbb{C}$ be as in the statement of theorem 5. For $\xi=(\xi_i)_{1\le i\le n}\in\mathbb{C}^n$, we will denote $|\xi|=\left(\sum_i|\xi_i|^2\right)^{1/2}$ (the usual norm) and $N(\xi)=\left(\sum_i\xi_i^2\right)^{1/2}$ with principal value of the square root. Notice that for $\xi\in\mathbb{R}^n$, $N(\xi)=N(\overline{\xi})=|\xi|$.

In this section, we choose some quantities as follows:

1. let $\lambda>0$ large enough (for instance $\lambda=4(K+1)$) and $\xi_0=(\lambda,0,\dots,0)\in\mathbb{R}^n$,
2. let $\delta>0$ small enough such that for every $\xi\in\mathbb{R}^n$ and $x\in\mathbb{R}^n$, $|\xi-\xi_0|<\delta$ and $|x|<\delta$ implies $N(\xi+\xi_0+ix)\in\mathcal{C}$,
3. let $\chi\in C_c^\infty(B(0,\delta))$ such that $0\le\chi\le1$ and $\chi\equiv1$ on a neighborhood of 0, say $B(0,\delta_2)$,
4. finally, for h small enough we define

$$\epsilon(h) := T\sup_{|\xi|<\delta,|x|<\delta} h\left|\rho\left(\frac{N(\xi+\xi_0+ix)}{h}\right)\right|, \tag{5}$$

Under these assumptions, we will state some upper and lower bounds on the following function:

$$g_h(t,x) := \int_{\mathbb{R}^n}\chi(\xi-\xi_0)\mathrm{e}^{-(\xi-\xi_0)^2/2h+\mathrm{i}x\xi/h-t\overline{\rho}(|\xi|/h)}\,\mathrm{d}\xi. \tag{6}$$

Proposition 15. *Under the above assumptions, we have uniformly in $0 \le t \le T$ and $|x|$ small enough*

$$g_h(t,x) = (2\pi h)^{n/2} \mathrm{e}^{\mathrm{i}x\xi_0/h - x^2/2h - t\overline{\rho}\left(\frac{N(\xi_0 - \mathrm{i}x)}{h}\right) + O\left(\frac{\epsilon(h)^2}{h}\right)}(1 + O(h + \epsilon(h))),$$

in the limit $h \to 0^+$.

Proof. With $\rho_{t,h}(\xi) := -t\overline{\rho}(N(\overline{\xi}))$ and with the notations of [16, §3.2], $g_h(t,x) = I_{t,h,1}(x)$. Then apply [16, Proposition 3.5] adapted in dimension n [16, §4.3]. □

Proposition 16. *Let $\eta > 0$ and $N \in \mathbb{N}$. Under the above assumptions, we have uniformly in $0 \le t \le T$ and $|x| > \eta$*

$$|g_h(t,x)| \le \frac{C}{|x|^N} \mathrm{e}^{-c/h}.$$

Proof. Apply [16, Proposition 3.7] adapted in dimension n [16, §4.3] with $\rho_{t,h}(\xi) = -t\overline{\rho}(N(\overline{\xi}))$. Note that with the notations of this theorem, $\epsilon(h)/h = o(c/h)$. □

With these estimates, we can prove theorem 5.

Proof of theorem 5. Let $\epsilon > 0$ as in the statement of theorem 5, and let $\epsilon' > 0$ small enough (depending on ϵ) to be chosen later.

Step 1: Observability inequality. — As in the proof of theorem 4 (and see [9, Theorem 2.44]), the exact null-controllability of the system $(\partial_t + \rho(|D_x|))f = \mathbb{1}_\omega u$ in time T is equivalent to the following observability inequality: for every $g_0 \in L^2(\mathbb{R})$, the solution g of

$$\partial_t g(t,x) + \overline{\rho}(|D_x|) g(t,x) = 0, \quad g(0,\cdot) = g_0, \tag{7}$$

satisfies

$$\|g(T,\cdot)\|_{L^2(\mathbb{R}^n)} \le C\|g\|_{L^2((0,T)\times\omega)}. \tag{8}$$

Throughout this proof, c and C denote constants that can change from line to line.

Step 2: Choice of test functions. — We want to find a lower bound on $\mathrm{Leb}(\omega \cap B(x,L))$ by testing the observability inequality on g_h defined in eq. (6). Since the equation (E_ρ) is invariant by translation, we may assume that $x = 0$. Notice that g_h satisfies

$$g_h(t,x) = h^n \int_{\mathbb{R}^n} \chi(h\xi - \xi_0) \mathrm{e}^{-(h\xi - \xi_0)^2/2h + \mathrm{i}x\xi - t\overline{\rho}(|\xi|)} \,\mathrm{d}\xi.$$

Thus, g_h is a solution to the heat-like equation (7).

Step 3: Lower bound on g_h. — Let $R \in (0,1)$ be such that for every $A > 2K$, $\overline{B(A, AR)} \subset \mathcal{C}$. According to Harnack's inequality [8, Chapter X, Theorem 2.14], if $A > 2K$ and $|\mu| < AR$, we have

$$\operatorname{Re}\rho\,(A+\mu) \le \frac{AR + |\mu|}{AR - |\mu|} \operatorname{Re}\rho\,(A).$$

Hence, if $\delta' < R$, for every $|z| < \delta'$ and $h > 0$ small enough, it follows that

$$\operatorname{Re}\rho\left(\frac{\lambda + z}{h}\right) \le \frac{\lambda R + \delta'}{\lambda R - \delta'} \operatorname{Re}\rho\left(\frac{\lambda}{h}\right).$$

We choose δ' such that $(\lambda R + \delta')/(\lambda R - \delta') < 1 + \epsilon'$. Reducing δ if necessary, we may assume that $\delta < \eta$ (the one from proposition 16) and that for $\xi, x \in \mathbb{R}^n$ with $|\xi| < \delta$ and $|x| < \delta$, $|N(\xi_0 + \xi + \mathrm{i}x) - N(\xi_0)| < \delta'$. In this case, we have

$$\operatorname{Re}\rho\left(\frac{N(\xi_0 + \xi + \mathrm{i}x)}{h}\right) \le (1+\epsilon')\operatorname{Re}\rho\left(\frac{N(\xi_0)}{h}\right) = (1+\epsilon')\operatorname{Re}\rho\left(\frac{\lambda}{h}\right).$$

Plugging this into the asymptotic of proposition 15, we deduce that for h small enough

$$\begin{aligned}\|g_h(T,\cdot)\|^2_{L^2(\mathbb{R}^n)} &\geq \|g_h(T,\cdot)\|^2_{L^2(|x|<\delta)}\\ &\geq ch^n\int_{|x|<\delta} \mathrm{e}^{-x^2/h-2T\operatorname{Re}\rho(N(\xi_0+\xi-\mathrm{i}x)/h)+O(\epsilon(h)^2/h)}\,\mathrm{d}x\\ &\geq ch^n\int_{|x|<\delta} \mathrm{e}^{-x^2/h-2T(1+\epsilon')\operatorname{Re}\rho(\lambda/h)+O(\epsilon(h)^2/h)}\,\mathrm{d}x\\ &\geq ch^{3n/2}\mathrm{e}^{-2T(1+\epsilon')\operatorname{Re}\rho(\lambda/h)+O(\epsilon(h)^2/h)}.\end{aligned}$$

Since we assumed that $|\operatorname{Im}\rho|\leq C\operatorname{Re}\rho$, we get that $\epsilon(h)\leq h(1+\epsilon')(1+C)T\operatorname{Re}\rho(\lambda/h)$ (see the definition of ϵ eq. (5)). We deduce that

$$\|g_h(T,\cdot)\|^2_{L^2(\mathbb{R}^n)} \geq ch^{3n/2}\mathrm{e}^{-2T(1+\epsilon')\operatorname{Re}\rho(\lambda/h)\left(1+O(\epsilon(h))\right)}.$$

Finally, for h small enough, $(1+\epsilon')(1+O(\epsilon(h)))<1+2\epsilon'$. We get

$$\|g_h(T,\cdot)\|^2_{L^2(\mathbb{R}^n)} \geq ch^{3n/2}\mathrm{e}^{-2T(1+2\epsilon')\operatorname{Re}\rho(\lambda/h)}. \tag{9}$$

Step 4: Upper bound on g_h. — Recall that for h small enough, $|\xi|<\delta$ and $|x|<\delta$, $\operatorname{Re}\rho(N(\xi_0+\xi+\mathrm{i}x)/h)\geq 0$. Hence, the asymptotics stated in proposition 15 imply the upper bound

$$|g_h(t,x)|\leq Ch^{n/2}\mathrm{e}^{-x^2/2h+O(\epsilon(h)/h)}.$$

Moreover, according to proposition 16, for every $h>0$ small enough and for every $0<r<\delta$,

$$\begin{aligned}\|g_h\|^2_{L^2((0,T)\times\omega)} &\leq \|g_h\|^2_{L^2((0,T)\times\{\delta<|x|\})} + \|g_h\|^2_{L^2((0,T)\times\{r<|x|<\delta\})} + \|g_h\|^2_{L^2((0,T)\times\{|x|<r\}\cap\omega)}\\ &\leq C\mathrm{e}^{-c/h} + Ch^n\mathrm{e}^{-r^2/h} + Ch^n\operatorname{Leb}(B(0,r)\cap\omega).\end{aligned}$$

Reducing δ if necessary, we may assume that $r^2<c$, and we can drop the first term of the right-hand side:

$$\|g_h\|^2_{L^2((0,T)\times\omega)} \leq Ch^n\mathrm{e}^{-r^2/h} + Ch^n\operatorname{Leb}(B(0,r)\cap\omega). \tag{10}$$

Step 5: Conclusion. — If the observability inequality (8) holds, according to eqs. (9) and (10), there exist $c,C>0$ such that for any $r>0$ and $h>0$ small enough:

$$ch^{n/2}\mathrm{e}^{-2T(1+2\epsilon')\operatorname{Re}\rho(\lambda/h)}\leq C\mathrm{e}^{-r^2/h}+C\operatorname{Leb}(B(0,r)\cap\omega).$$

Since $\rho(\xi)=o(\xi)$ and $\sqrt{h_r 2T(1+\epsilon)\operatorname{Re}\rho(\lambda/h_r)}\leq r$, we get that $h_r\to 0$ as $r\to 0$. Hence, for r small enough, we can apply the previous inequality with $h=h_r$, which gives us

$$\begin{aligned}ch_r^{n/2}\mathrm{e}^{-2T(1+2\epsilon')\operatorname{Re}\rho(\lambda/h_r)} &\leq C\mathrm{e}^{-r^2/h_r}+C\operatorname{Leb}(B(0,r)\cap\omega)\\ &\leq C\mathrm{e}^{-2T(1+\epsilon)\operatorname{Re}\rho(\lambda/h_r)}+C\operatorname{Leb}(B(0,r)\cap\omega).\end{aligned}$$

This gives us

$$\operatorname{Leb}(B(0,r)\cap\omega)\geq ch_r^{n/2}\mathrm{e}^{-2T(1+2\epsilon')\operatorname{Re}\rho(\lambda/h_r)}-c\mathrm{e}^{-2T(1+\epsilon)\operatorname{Re}\rho(\lambda/h_r)}$$

Now, since $\ln(\xi)=o(\operatorname{Re}\rho(\xi))$, we get $h_r^{n/2}\geq C\mathrm{e}^{-2T\epsilon'\operatorname{Re}\rho(\lambda/h_r)}$. For small enough h_r (equivalently, small enough $r>0$), this gives us

$$\operatorname{Leb}(B(0,r)\cap\omega)\geq c\mathrm{e}^{-2T(1+3\epsilon')\operatorname{Re}\rho(\lambda/h_r)}-c\mathrm{e}^{-2T(1+\epsilon)\operatorname{Re}\rho(\lambda/h_r)}.$$

If we choose ϵ' small enough so that $3\epsilon' < \epsilon$, the second term of the right-hand side is negligible, and we get for h small enough

$$\begin{aligned}\mathrm{Leb}(B(0,r)\cap\omega) &\geq ce^{-2T(1+3\epsilon')\operatorname{Re}\rho(\lambda/h_r)}\\ &\geq ce^{-2T(1+\epsilon)\operatorname{Re}\rho(\lambda/h_r)}.\end{aligned}$$

Dividing this inequality by $\mathrm{Leb}(B(x,r))$ gives the claimed inequality. □

4. Examples of γ-thick sets

In this section, we construct non-trivial sets that are γ-thick for any given function $\gamma\colon (0,r_0]\to[0,1]$. This construction is based on Smith-Volterra-Cantor sets.

4.1. **Thickness of Smith-Volterra-Cantor sets.**

Let us first recall the definition of Smith-Volterra-Cantor sets.

Definition 17 (Smith-Volterra-Cantor sets)**.** Let $(\tau_n)_{n\in\mathbb{N}}$ be a sequence of real numbers such that $0<\tau_n<1$. For $n\in\mathbb{N}$, let K_n be the closed subset of $[0,1]$, finite union of closed disjoint intervals, defined inductively by the following procedure.

- $K_0 := [0,1]$.
- If $K_n = \bigcup_k I_{nk}$, where the $(I_{nk})_k$ are disjoint closed intervals, remove from I_{nk} the middle part of size $\tau_n\,\mathrm{Leb}(I_{nk})$ and call the resulting sets[1] I'_{nk}. Then set $K_{n+1} := \bigcup_k I'_{nk}$.

Let $K := \bigcap_{n\in\mathbb{N}} K_n$. The set K is the Smith-Volterra-Cantor set associated to the sequence $(\tau_n)_n$.

At each step in the construction of a Smith-Volterra-Cantor set, we remove a subset of measure $(1-\tau_n)\,\mathrm{Leb}(K_n)$ from K_n (see definition 17), hence:

Proposition 18. *With the notations of definition 17,* $\mathrm{Leb}(K_n) = \prod_{k=0}^{n-1}(1-\tau_n)$ *and* $\mathrm{Leb}(K) = \prod_{n=0}^{+\infty}(1-\tau_n)$. *In particular,* $\mathrm{Leb}(K)>0$ *if and only if* $\sum_n \tau_n < +\infty$.

Our first result is some upper and lower bounds on the thickness of the complement of Smith-Volterra-Cantor sets.

Proposition 19. *Let* $(\tau_n)_n \in (0,1)^{\mathbb{N}}$. *Assume that* $\sum_n\tau_n<+\infty$. *Let K be the associated Smith-Volterra-Cantor set and* $\omega := \mathbb{R}\setminus K$. *There exist* $c>0$, $C>0$ *and* $r_1>0$ *such that for every* $0<r<r_1$,

$$\frac{1}{24}\sum_{k\geq \log_2(3\,\mathrm{Leb}(K)/r)}\tau_k \leq \inf_{x\in\mathbb{R}}\frac{\mathrm{Leb}(\omega\cap B(x,r))}{\mathrm{Leb}(B(x,r))}\leq 6\sum_{k\geq\log_2(\mathrm{Leb}(K)/4r)}\tau_k,$$

where $\log_2$ *is the base 2 logarithm* $\log_2(x) = \ln(x)/\ln(2)$.

With a more careful analysis in the proof below, it seems we could improve this inequality by replacing the $\log_2(3\,\mathrm{Leb}(K)/r)$ by $\log_2(\varkappa\,\mathrm{Leb}(K)/r)$ with some $\varkappa<3$. We do not know what the optimal $\varkappa$ is. We do not pursue this because we do not need such a sharp estimate. In the same spirit, the factors $1/24$ and 6 are not optimal either.

Proof. Remark that we only need to estimate $\mathrm{Leb}(\omega\cap B(x,r))$ for $x\in[0,1]$. Indeed, if for instance $x>1$, $\omega\cap B(x,r)$ contains $[x,x+r]$, hence $\mathrm{Leb}(\omega\cap B(x,r))/\mathrm{Leb}(B(x,r))\in[1/2,1]$.

Step 1: Notations and preliminary computations. — In this proof, we denote by I_{nk} the intervals that appears in the construction of K, as defined in definition 17. We denote the length of I_{nk} (which does not depend on k) by ℓ_n. We have

$$\ell_n = \frac{1-\tau_{n-1}}{2}\ell_{n-1}.$$

[1] I.e., if $I_{nk} = [a_{nk},b_{nk}]$, set $b'_{nk} := (a_{nk}(1+\tau_n)+b_{nk}(1-\tau_n))/2$ and $a'_{nk} = (a_{nk}(1-\tau_n)+b_{nk}(1+\tau_n))/2$, and finally $I'_{nk} := [a_{nk},b'_{nk}]\cup[a'_{nk},b_{nk}]$.

Notice that

$$\mathrm{Leb}(I_{nk} \cap \omega) = \mathrm{Leb}(I_{nk}) - \mathrm{Leb}(I_{nk} \cap K) = \ell_n \big(1 - \prod_{k \geq n} (1 - \tau_k)\big).$$

In addition, we can estimate the right-hand side in the following way

$$\begin{aligned} 1 - \prod_{k \geq n} (1 - \tau_k) &= 1 - \exp\Big(\sum_{k \geq n} \ln(1 - \tau_k)\Big) \\ &= 1 - \exp\Big(\sum_{k \geq n} -\tau_k (1 + o_k(1))\Big) \qquad \text{(because } \tau_k \to 0\text{)} \\ &= 1 - \exp\Big(-(1 + o_n(1)) \sum_{k \geq n} \tau_k\Big) \qquad \text{(because } \textstyle\sum \tau_k < +\infty\text{)} \\ &= 1 - \Big(1 - (1 + o_n(1)) \sum_{k \geq n} \tau_k\Big) \qquad \text{(because } \textstyle\sum_{k \geq n} \tau_k \xrightarrow[n \to \infty]{} 0\text{)} \\ &= (1 + o_n(1)) \sum_{k \geq n} \tau_k. \end{aligned}$$

Finally, multiplying by ℓ_n,

$$\mathrm{Leb}(I_{nk} \cap \omega) = (1 + o_n(1)) \ell_n \sum_{k \geq n} \tau_k. \tag{11}$$

Step 2: Lower bound when r is comparable to ℓ_n. — Let $r > 0$ and $n \in \mathbb{N}$ be such that $2\ell_n \leq r \leq 6\ell_n$. Let $x \in [0, 1]$.

If $\omega \cap B(x, r)$ contains an interval of length $\geq \ell_n/2$, $\mathrm{Leb}(\omega \cap B(x, r)) \geq \ell_n/2$.

If that is not the case, then, $\mathrm{distance}(x, K_n) < \ell_n/4$. Since $r \geq 2\ell_n$, this implies that $B(x, r)$ contains some I_{nk}. Hence, according to eq. (11),

$$\mathrm{Leb}(\omega \cap B(x, r)) \geq \mathrm{Leb}(\omega \cap I_{nk}) = (1 + o_n(1)) \ell_n \sum_{k \geq n} \tau_k.$$

Putting the two cases together, and assuming that n is large enough:

$$\inf_{x \in \mathbb{R}} \frac{\mathrm{Leb}(\omega \cap B(x, r))}{\mathrm{Leb}(B(x, r))} \geq \frac{\ell_n}{2r} \min\Big((1 + o_n(1)) \sum_{k \geq n} \tau_k, \frac{1}{2}\Big) \geq \frac{\ell_n}{4r} \sum_{k \geq n} \tau_k.$$

Since $r \leq 6\ell_n$,

$$\inf_{x \in \mathbb{R}} \frac{\mathrm{Leb}(\omega \cap B(x, r))}{\mathrm{Leb}(B(x, r))} \geq \frac{1}{24} \sum_{k \geq n} \tau_k, \tag{12}$$

this inequality being valid whenever n is large enough and when $2\ell_n \leq r \leq 6\ell_n$.

Step 3: Upper bound when r is comparable to ℓ_n. — Let $r > 0$ and $n \in \mathbb{N}$ be such that $\ell_n/3 \leq 2r \leq \ell_n$. Let $x \in [0, 1]$ in the middle of a I_{nk}, so that $B(x, \ell_n/2) = I_{nk}$. Then $B(x, r) \subset I_{nk}$, and according to eq. (11),

$$\mathrm{Leb}(\omega \cap B(x, r)) \leq \mathrm{Leb}(I_{nk} \cap \omega) = (1 + o_n(1)) \ell_n \sum_{k \geq n} \tau_k.$$

Since, $\ell_n/3 \leq 2r$, and assuming n is large enough

$$\inf_{x \in \mathbb{R}} \frac{\mathrm{Leb}(\omega \cap B(x, r))}{\mathrm{Leb}(B(x, r))} \leq (1 + o_n(1)) \frac{\ell_n}{2r} \sum_{k \geq n} \tau_k \leq 6 \sum_{k \geq n} \tau_k, \tag{13}$$

this inequality being valid whenever n is large enough and when $\ell_n/3 \leq 2r \leq \ell_n$.

Step 4: Solving the inequality $a\ell_n \leq r \leq b\ell_n$. — Let $r > 0$ and $0 < \varkappa < 1$. Set $n(r) := \lceil \log_2(\mathrm{Leb}(K)/r) \rceil$, where $\lceil \cdot \rceil$ is the ceiling function. We aim to prove that for r small enough, $\varkappa \ell_{n(r)} \leq r \leq 2\ell_{n(r)}$.

According to the definition of I_{nk}, $\ell_n = 2^{-n}\prod_{k=0}^{n-1}(1-\tau_k)$. Recall that $\mathrm{Leb}(K) = \prod_{k=0}^{+\infty}(1-\tau_k) > 0$. Define q_n by $\mathrm{Leb}(K)q_n = \prod_{k=0}^{n-1}(1-\tau_k)$, i.e., $q_n = \big(\prod_{k\geq n}(1-\tau_k)\big)^{-1}$. Then $q_n > 1$ and $q_n \to 1$ as $n \to +\infty$. With this notation, we have the equivalences

$$\begin{aligned}\varkappa\ell_n \leq r \leq 2\ell_n &\Leftrightarrow 2^{-n}\varkappa\,\mathrm{Leb}(K)q_n \leq r \leq 2^{1-n}\,\mathrm{Leb}(K)q_n\\ &\Leftrightarrow -n + \log_2(\varkappa\,\mathrm{Leb}(K)q_n) \leq \log_2(r) \leq 1 - n + \log_2(\mathrm{Leb}(K)q_n)\\ &\Leftrightarrow \log_2\Big(\frac{\mathrm{Leb}(K)}{r}\Big) + \log_2(\varkappa) + \log_2(q_n) \leq n \leq \log_2\Big(\frac{\mathrm{Leb}(K)}{r}\Big) + 1 + \log_2(q_n). \quad (14)\end{aligned}$$

According to the definition of the ceiling function,

$$\log_2\Big(\frac{\mathrm{Leb}(K)}{r}\Big) \leq n(r) < \log_2\Big(\frac{\mathrm{Leb}(K)}{r}\Big) + 1.$$

Since $\log_2(\varkappa) < 0$, $\log_2(q_n) > 0$ and $q_n \to 1$, the inequalities on the right-hand side of eq. (14) are satisfied for $n = n(r)$ and small enough $r > 0$. Hence, for r small enough, $\varkappa\ell_{n(r)} \leq r \leq 2\ell_{n(r)}$, as claimed.

Step 5: Conclusion. — Applying the previous step with $\varkappa = 2/3$, and replacing r by $r/3$, we see that lower bound (12) holds for $n = \big\lceil \log_2\big(\frac{3\,\mathrm{Leb}(K)}{r}\big)\big\rceil$ when r is small enough. Hence, the lower-bound stated in proposition 19 holds.

Applying step 4 with $\varkappa = 2/3$ and r replaced by $4r$, we get that the upper bound (13) holds with $n = \big\lceil \log_2\big(\frac{\mathrm{Leb}(K)}{4r}\big)\big\rceil$ and r small enough, which gives the stated upper bound. □

4.2. **Construction of γ-thick sets.** Let $\gamma\colon (0, r_0] \to [0, 1]$ be such that $\gamma(r) \to 0$ as $r \to 0$.[2] Let us explain how we can use proposition 19 to construct a set that is thick with respect to the function γ, or more precisely $\gamma_{|(0,r_1]}$ for some small enough r_1 (this does not matter, because considering a γ-thick set or a $\gamma_{|(0,r_1]}$-thick set is the same as far as our theorems are concerned).

First, we prove that it is sufficient to treat the case where γ is increasing. Indeed, set $\gamma_1(r) := \sup_{(0,r]}\gamma$, i.e., the smallest non-decreasing function that is larger or equal than γ. Finally, we set $\gamma_2 := \gamma_1 + \phi$ where ϕ is a small nonnegative increasing function such that $\lim_{r\to 0}\phi(r) = 0$. This function γ_2 is such that $\gamma_2(r) \to 0$ as $r \to 0$, γ_2 is increasing and $\gamma_2(r) \geq \gamma(r)$. Hence, a set that is thick with respect to γ_2 will also be thick with respect to γ. From now on, we assume that γ is increasing.

We are looking for a sequence $(\tau_k)_k \in (0,1)^{\mathbb{N}}$ such that $\sum \tau_k < +\infty$ and such that the left-hand side of the thickness estimate in proposition 19 is larger than $\gamma(r)$, at least for small enough r. Moreover, if K is the Smith-Volterra-Cantor set associated to $(\tau_k)_k$ we will look for such a sequence such that $\mathrm{Leb}(K) = 1/3$. In other words, we want $0 < \tau_k < 1$ and:

$$\mathrm{Leb}(K) = \prod_{k=0}^{+\infty}(1-\tau_k) = \frac{1}{3}, \tag{15}$$

$$\exists r_0 > 0,\ \forall 0 < r \leq r_0,\ \gamma(r) \leq \frac{1}{24}\sum_{k\geq \log_2(1/r)}\tau_k. \tag{16}$$

The thickness condition (16) imposes $\sum_{k\geq n}\tau_k \geq 24\sup_{[2^{-n-1},2^{-n})}\gamma = 24\gamma(2^{-n})$ since γ is increasing. Motivated by this, we set $\tau_n := 24(\gamma(2^{-n}) - \gamma(2^{-n-1}))$ when it is defined, i.e., for $n \geq n_0 := \lceil \log_2(1/r_0)\rceil$. Notice that since γ is increasing, we do have $\tau_k > 0$. For $n \geq n_0$, we have $\sum_{k\geq n}\tau_k =$

[2] The case $\gamma(r) \to 0$ as $r \to 0$ is the interesting one. In fact, we claim that if $\limsup_{r\to 0}\gamma(r) > 0$ and $\omega \subset \mathbb{R}^n$ is thick with respect to γ, then $\mathbb{R}^n \setminus \omega$ is negligible. Indeed, in this case, the definition of γ-thickness tells us that for all $x \in \mathbb{R}^n$, $\limsup_{r\to 0}\mathrm{Leb}(\omega \cap B(x,r))/\mathrm{Leb}(B(x,r)) > 0$. On the other hand, Lebesgue's differentiation theorem applied to $\mathbb{1}_\omega$ implies that for almost every $x \notin \omega$, $\lim_{r\to 0}\mathrm{Leb}(\omega \cap B(x,r))/\mathrm{Leb}(B(x,r)) = 0$, which is only possible if $\mathbb{R}^n \setminus \omega$ is negligible.

$24(\gamma(2^{-n})-\lim_0\gamma) = 24\gamma(2^{-n}) < +\infty$. For $n < n_0$, we choose some arbitrary value for τ_n, for instance $\tau_n = 1/2$. This sequence (τ_k) satisfies the thickness condition (16) by construction.

Of course, we might not have the measure condition (15), and for some k, we might not even have $\tau_k < 1$. But we can tweak the sequence $(\tau_k)_k$ to ensure these properties. Notice that if we change a finite number of τ_k, the thickness condition (16) still holds (with a smaller r_0). There are a finite number of τ_k that are larger or equal than 1 (if any), and we can set them to, e.g., $1/2$. Next, if $\mathrm{Leb}(K) > 1/3$, increase τ_0 to reduce $\mathrm{Leb}(K)$. And if $\mathrm{Leb}(K) < 1/3$, choose N so that $\prod_{k=N+1}^{+\infty}(1-\tau_k) > 1/3$, and decrease $\tau_0, \dots, \tau_N$ to increase $\mathrm{Leb}(K)$.

Let us end this construction by noticing that it is almost optimal in the following sense: according to the right-hand side of the thickness estimate in proposition 19, the set $\omega = \mathbb{R} \setminus K$ is such that for r small enough,

$$\begin{aligned}\inf_{x\in\mathbb{R}} \frac{\mathrm{Leb}(B(x,r)\cap\omega)}{\mathrm{Leb}(B(x,r))} &\leq 6\sum_{k\geq \log_2(1/12r)} \tau_k \\ &= 6\times 24\gamma\left(2^{-\lceil \log_2(1/12r)\rceil}\right) \\ &\leq 144\gamma(12r).\end{aligned}$$

We summarize this construction in the following proposition:

Proposition 20. *Let $\gamma : (0, r_0] \to [0,1]$ be such that $\gamma(r) \to 0$ as $r \to 0$. There exists a set $\omega \subset \mathbb{R}$ such that $\mathrm{Leb}(\mathbb{R}\setminus\omega) > 0$ and such that for every r small enough,*

$$\gamma(r) \leq \inf_{x\in\mathbb{R}} \frac{\mathrm{Leb}(B(x,r)\cap\omega)}{\mathrm{Leb}(B(x,r))}.$$

Moreover, if γ is increasing, we can choose $\omega = \mathbb{R}\setminus K$, where K is the Smith-Volterra-Cantor set associated to a sequence $(\tau_n)_n$ such that for n large enough, $\tau_n = 24(\gamma(2^{-n}) - \gamma(2^{-n-1}))$, in which case, for small enough r,

$$\inf_{x\in\mathbb{R}} \frac{\mathrm{Leb}(B(x,r)\cap\omega)}{\mathrm{Leb}(B(x,r))} \leq 144\gamma(12r).$$

This proposition constructs γ-thick sets only in dimension 1. In higher dimension, we can prove that if $\omega_1 \subset \mathbb{R}$ is γ-thick, then $\omega := \omega_1 \times \mathbb{R}^{n-1}$ is thick with respect to $a_n\gamma(b_n\cdot)$, for some universal constants a_n, b_n that depends only on n. Indeed, let $b_n > 0$ such that $[-b_n, b_n]^n \subset B(0,1)$. Then,

$$\begin{aligned}\mathrm{Leb}(B(x,r)\cap\omega) &\geq \mathrm{Leb}([x-b_nr, x+b_nr]^n\cap\omega) && ([x-b_nr,x+b_nr]^n \subset B(x,r)) \\ &\geq \mathrm{Leb}([x-b_nr,x+b_nr]\cap\omega_1)(2b_nr)^{n-1} && ((\textstyle\prod_i A_i)\cap(\prod_i B_i) = \prod_i(A_i\cap B_i)) \\ &\geq \gamma(b_nr)(2b_nr)^n && (\omega_1 \text{ is thick with respect to } \gamma).\end{aligned}$$

Thus,

$$\frac{\mathrm{Leb}(B(x,r)\cap\omega)}{\mathrm{Leb}(B(x,r)} \geq \frac{(2b_n)^n}{\mathrm{Leb}(B(0,1))}\gamma(b_nr).$$

Appendix A. Heat-like equation on the torus

The previous results are stated for equations posed on the whole space. But we can adapt these results when the equation is posed on the torus $\mathbb{T}^n := (\mathbb{R}/2\pi\mathbb{Z})^n$. The equation is written in the same way:

$$(E_\rho^{\mathbb{T}}) \qquad \begin{cases}\partial_t f(t,x) + \rho(|D_x|)f(t,x) = \mathbb{1}_\omega u(t,x), & (t,x)\in\mathbb{R}_+^*\times\mathbb{T}^n, \\ f(0,\cdot) = f_0 \in L^2(\mathbb{T}^n),\end{cases}$$

where $\rho(|D_x|)$ is again defined with the functional calculus, that is to say, for $f\in L^2(\mathbb{T}^n)$ and $k\in\mathbb{Z}^n$, denoting the k-th Fourier coefficient of f by $c_k(f)$, we define $\rho(|D_x|)f$ by $c_k(\rho(|D_x|)f) = \rho(|k|)c_k(f)$.

The strongly dissipative case (e.g. $\rho(\xi) = \xi^s$ with $s > 1$) for these equations on bounded domains or compact manifold is well known, by combining Burq and Moyano's estimate [7, Theorem 1] and Lebeau and Robbiano's method, as stated by Duyckaerts and Miller [10, Theorem 6.1] (see also references in those two articles).

For the weakly dissipative case, we have the following straightforward adaptations of definition 3 and theorems 4 and 5.

Definition 21. Given some $r_0 > 0$ and a function $\gamma : (0, r_0] \to [0,1]$, a set $\omega \subset \mathbb{T}^n$ is said to be *thick relatively to* γ (or γ-thick) when it is measurable and satisfies that for every $r \in (0, r_0]$ and $x \in \mathbb{T}^n$,

$$\operatorname{Leb}(\omega \cap B(x,r)) \geq \gamma(r) \operatorname{Leb}(B(x,r)).$$

Theorem 22 (theorem 4 in the torus)**.** *Let $\rho : [0, +\infty) \to \mathbb{C}$ and γ_ρ be as in theorem 4, and $\omega \subset \mathbb{T}^n$ be γ_ρ-thick. For every $T > 0$, the parabolic equation ($E_\rho^{\mathbb{T}}$) is null-controllable from ω in time T.*

Sketch of the proof. Egidi and Veselić proved a version of Kovrijkine's estimate for functions defined on the torus [13, Theorem 2.1]. The proof of theorem 22 is a copy-paste of the one of theorem 4, where we replace Kovrijkine's estimate (4) by the aforementionned version on the torus. □

Theorem 23 (theorem 5 on the torus)**.** *Let $K > 0$, $\mathcal{C} \subset \mathbb{C}$, $\rho : \mathcal{C} \cup \mathbb{R}_+ \to \mathbb{C}$ and h_r be as in theorem 5. Let $T > 0$ and $\omega \subset \mathbb{T}^d$ be measurable. Assume that the parabolic equation ($E_\rho^{\mathbb{T}}$) is null controllable from ω in time $T > 0$.*

There exists $\lambda > 0$, $r_0 > 0$ and $c > 0$ such that for every $\epsilon > 0$, and for every function $r \in (0, r_0] \mapsto h_r \in \mathbb{R}_+^*$ that satisfy

$$\forall r \in (0, r_0], \quad \sqrt{h_r 2T(1+\epsilon) \operatorname{Re} \rho\left(\frac{\lambda}{h_r}\right)} \leq r, \tag{17}$$

then, there exists $r_1 \in (0, r_0)$ such that for every $0 < r < r_1$, and $x \in \mathbb{R}^n$, we have

$$\frac{\operatorname{Leb}(\omega \cap B(x,r))}{\operatorname{Leb}(B(x,r))} \geq c r^{-n} \exp\left(-2T(1+\epsilon) \operatorname{Re} \rho\left(\frac{\lambda}{h_r}\right)\right).$$

Sketch of the proof. Consider g_h as defined by eq. (6) and

$$g_{h\text{per}}(t,x) := \sum_{k \in \mathbb{Z}^n} g_h(t, x + 2\pi k).$$

We can check that $g_{h\text{per}}$ is a solution of the adjoint equation $\partial_t g(t,x) + \overline{\rho}(|D_x|) g(t,x) = 0$ on the torus (see [16, §4.2]). Moreover, the terms for $k \neq 0$ are exponentially small. Hence, when estimating the left-hand side $\|g_{h\text{per}}(T,\cdot)\|_{L^2}$ and the right-hand side $\|g_{h\text{per}}\|_{L^2([0,T]\times\omega)}$ of the observability inequality, only the term for $k = 0$ matters. Thus, we can do all the computations of the proof of theorem 5 with $g_{h\text{per}}$ instead of g_h. □

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